Math 246C Lecture Notes

Professor: Michael Hitrik
Scribe: Daniel Raban

Contents

1 Introduction to Riemann Surfaces

In this course, we will study two main topics:

1. Introduction to Riemann surfaces.

2. Introduction to several complex variables.

1.1 Complex charts and atlases

Definition 1.1. Let X be a Hasudorff topological space. A **complex chart** on X is a homeomorphism $\varphi : U \to V$, where $U \subseteq X$ and $V \subseteq \mathbb{C}$ are open. Two charts $\varphi_1 : U_1 \to V_1$ and $\varphi_2 : U_2 \to V_2$ are called **compatible** if $U_1 \cap U_2 = \varnothing$ or the **transition map** $\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$ is holomorphic. A **complex atlas** on X is a collection of pairwise compatible charts $\{\varphi_\alpha : U_\alpha \to V_\alpha\}_{\alpha \in A}$ such that $X = \bigcup_{\alpha \in A} U_\alpha$.

Remark 1.1. It follows that $\varphi_2 \circ \varphi_1^{-1}$ is a holomorphic diffeomorphism.

Proposition 1.1. *Let $\mathscr{A} = \{\varphi_\alpha : U_\alpha \to V_\alpha\}$ be a complex atlas for X. The collection $\widehat{\mathscr{A}} = \{\varphi : U \to V : \varphi$ is a chart on X, φ and φ_α are compatible $\forall \alpha\}$ is a complex atlas for X, $\mathscr{A} \subseteq \widehat{\mathscr{A}}$, and this atlas is maximal. If $\mathscr{A} \subseteq \mathscr{B}$, then $\mathscr{B} \subseteq \widehat{\mathscr{A}}$*

Proof. We only need to check that $\widehat{\mathscr{A}}$ is an atlas. Let $\varphi_1 : U_1 \to V_1$, $\varphi_2 : U_2 \to V_2$ be charts in $\widehat{\mathscr{A}}$, and check that $\varphi_2 \circ \varphi_1^{-1}$ is holomorphic: Let $z \in \varphi_1(U_1 \cap U_2)$ and let $\varphi_\alpha : U_\alpha \to V_\alpha$ be a chart in $\mathscr{A}$ such that $\varphi_1^{-1}(z) \in U_\alpha$. Then $\varphi_1(U_1 \cap U_2 \cap U_\alpha)$ is a neighborhood of z, and $\varphi_2 \circ \varphi_1^{-1}$:

$$\varphi_1(U_1 \cap U_2 \cap U_\alpha) \xrightarrow{\varphi_\alpha \circ \varphi_1^{-1}} \varphi_\alpha(U_1 \cap U_2 \cap U_\alpha) \xrightarrow{\varphi_2 \circ \varphi_\alpha^{-1}} \varphi_2(U_1 \cap U_2 \cap U_\alpha)$$

is holomorphic. □

Remark 1.2. An atlas of the form $\widehat{\mathscr{A}}$ is called **maximal**.

Definition 1.2. We say that atlases $\mathscr{A} = \{\varphi_\alpha : U_\alpha \to V_\alpha\}, \mathscr{B} = \{\varphi'_\beta : U'_\beta \to V'_\beta\}$ are **equivalent** if $\varphi_\alpha, \varphi'_\beta$ are compatible for all α, β.

Remark 1.3. $\mathscr{A}$ is equivalent to $\mathscr{B}$ iff $\widehat{\mathscr{A}} = \widehat{\mathscr{B}}$.

1.2 Riemann surfaces

Definition 1.3. A **complex structure** on X is given by a maximal atlas on X. A **Riemann surface** is a connected, Hausdorff topological space equipped with a complex structure.

Example 1.1. Let $\Omega \subseteq \mathbb{C}$ be open and connected. Then Ω is a Riemann surface when equipped with the atlas $\{1 : \Omega \to \Omega\}$.

Example 1.2. The Riemann sphere $\widehat{\mathbb{C}} \cup \{\infty\}$ with the usual topology is a Riemann surface. Let $U_1 = \mathbb{C}$, $U_2 = \widehat{\mathbb{C}} \setminus \{0\}$ be open, and define the charts $\varphi_1 : U_1 \to \mathbb{C}$ sending $z \mapsto z$ and $\varphi_2 : U_2 \to \mathbb{C}$ send

$$\varphi_2(z) = \begin{cases} 1/z & z \in \mathbb{C} \setminus \{0\} \\ 0 & z = \infty. \end{cases}$$

To check compatibility, $\varphi_2 \circ \varphi_1^{-1}(z) = 1/z$ as a function from $\mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$. The atlas $(\varphi_j, U_j)_{j=1,2}$ gives rise to a Riemann surface structure on $\widehat{\mathbb{C}}$.

Example 1.3 (complex tori)**.** Let $e_1, e_2 \in \mathbb{C}$ be $\mathbb{R}$-linearly independent, and let Λ be the lattice $\Lambda = \{me_1 + ne_2 : m, n \in \mathbb{Z}\} \subseteq \mathbb{C}$. We have the equivalence relation $z \sim w$ if $z - w \in \Lambda$ and let $\mathbb{C}/\Lambda = z + \Lambda : z \in \mathbb{C}\}$ be the collection of equivalence classes. We have the projection map $\pi : \mathbb{C} \to \mathbb{C}/\Lambda$ sending $z \mapsto z + \Lambda$. We equip $\mathbb{C}/\Lambda$ with the strongest topology such that π is continuous: $O \subseteq \mathbb{C}/\Lambda$ is open if $\pi^{-1}(O) \subseteq \mathbb{C}$ is open. Then $\mathbb{C}/\Lambda$ is connected and compact. Compactness follows from $\mathbb{C}/\Lambda = \pi(\{te_1 + se_2 : 0 \leq t, s \leq 1\})$.

We claim that π is an open map. Let $V \subseteq \mathbb{C}$ be open. Then $\pi(V) \subseteq \mathbb{C}/\Lambda$ is open iff $\pi^{-1}(\pi(V)) \subseteq \mathbb{C}$ is open. This is $\pi^{-1}(\pi(V)) = \{z \in \mathbb{C} : \pi(z) \in \pi(V)\} = \bigcup_{\zeta \in \Lambda} (\zeta + V)$.

We need complex charts on $\mathbb{C}/\Lambda$: Let $V \subseteq \mathbb{C}$ be open such that no 2 distinct points of V are equivalent under Λ. Then $\pi|_V : V \to \pi(V) = U$ is a homeomorphism, and $\varphi = (\pi_V)^{-1}$ is a chart.

2 Holomorphic Curves in $\mathbb{C}^2$ and Holomorphic Functions on Riemann Surfaces

2.1 Holomorphic curves in $\mathbb{C}^2$

Last time, we were discussing complex tori.

Example 2.1 (complex tori)**.** We have $X = \mathbb{C}/\Lambda$, where Λ is a lattice. We have a natural quotient map $\pi : \mathbb{C} \to \mathbb{C}/\Lambda$. Let V_1, V_2 be the images of two charts $\varphi_i : U_i \to V_i$, $i = 1, 2$. Consider $\varphi_2 \circ \varphi_1^{-1}(z) =: \psi(z)$. Then for $z \in \varphi_1(U_1 \cap U_2)$, $\pi|_{V_2}(\psi(z)) = \pi|_{V_1}(z)$, so $\psi(z) - z \in \Lambda$. Since Λ is discrete, $\psi(z) - z$ is locally constant. So it is holomorphic.

Here is another natural example of a Riemann surface.

Example 2.2 (holomorphic curves in $\mathbb{C}^2 = \mathbb{C}^2_{z,w}$)**.** Let $\Omega \subseteq \mathbb{C}^2$ be open, and let $f \in \operatorname{Hol}(\Omega)$; that is, $f \in C^1(\Omega)$, and $f(z, w)$ is separately holomorphic: $z \mapsto f(z, w)$ is holomorphic for all w and $w \mapsto f(z, w)$ is holomorphic for all z. We have the Cauchy-Riemann equations

$$\frac{\partial f}{\partial \overline{z}}(z, w) = 0, \qquad \frac{\partial f}{\partial \overline{w}}(z, w) = 0.$$

Assume that $(\frac{\partial f}{\partial z}, \frac{\partial f}{\partial w}) \neq 0$ for all $(z, w) \in f^{-1}(\{0\})$.

We claim that $X = f^{-1}(\{0\})$ is a (possibly disconnected) Riemann surface. Let $(z_0, w_0) \in X$. If $f'_w(z_0, w_0) \neq 0$, then by the holomorphic implicit function theorem (which we will prove), there exist an open neighborhood $V \subseteq \mathbb{C}^2$ of (z_0, w_0), $z_0 \in U \subseteq \mathbb{C}$, and $g \in \operatorname{Hol}(U)$ such that $X \cap V = \{(z, g(z)) : z \in U\}$. So the projection $\pi_z : X \cap V \to U$ sending $(z, w) \mapsto z$ is a chart. Similarly, if $f'_z(z_0, w_0) \neq 0$, we have locally near (z_0, w_0): $X \cap V = \{(h(w), w)\}$, where h is holomorphic. So the projection $\pi_w : X \cap V \to \mathbb{C}$ is a chart. Compatibility of charts is the following diagram:

$$\begin{tikzcd} X \arrow[r, "\pi_w"] \arrow[d, "\pi_z"'] & U_w \\ U_z \arrow[ur, "\pi_w \circ \pi_z^{-1}"'] & \end{tikzcd}$$

Theorem 2.1 (holomorphic implicit function theorem)**.** *Let $f(z, w) : \mathbb{C}^2 \to \mathbb{C}$ be holomorphic near $(0, 0) \in \mathbb{C}^2$ with $f'(a, b) \neq 0$. Then $f = 0$ determines a holomorphic map $\varphi : \mathbb{C} \to \mathbb{C}$ in a neighborhood of (a, b).*

Proof. Let $f(z, w)$ be holomorphic near $(0, 0) \in \mathbb{C}^2$ with $f(0, 0) = 0$ and $f'_w(0, 0) \neq 0$. Choose $r > 0$ so that $w \mapsto f(0, w)$ is holomorphic when $|w| < 2r$ and $f(0, w) \neq 0$ when $0 < |w| < 2r$. Then choose $\delta > 0$ such that f is holomorphic when $|w| < 3r/2$, $|z| < \delta$ and such that $f(z, w) \neq 0$ when $|w| = r$, $|z| < \delta$. By the argument principle, for $|z| < \delta$,

$$|\{w \in D(0, r) : f(z, w) = 0\}| = \frac{1}{2\pi i} \int_{|w|=r} \frac{f'_w(z, w)}{f(z, w)} \, dw,$$

where the right hand side is holomorphic in z. So for all z with $|z| < \delta$, the equation $f(z,w) = 0$ has exactly 1 root $w = w(z)$ in $D(0,r)$. Write

$$w(z) = \frac{1}{2\pi i}\int_{|w|=r} \frac{wf'_w(z,w)}{f(z,w)}\,dw, \qquad |z| < \delta$$

by the residue theorem. ☐

2.2 Holomorphic functions on Riemann surfaces

Definition 2.1. Let X be a Riemann surface equipped with an atlas $\{\varphi_\alpha : U_\alpha \to V_\alpha\}$. We say that $f : X \to \mathbb{C}$ is **holomorphic** if for all α, $f \circ \varphi_\alpha^{-1} \in \mathrm{Hol}(V_\alpha)$. Let Y be a Riemann surface equipped with an atlas $\{\varphi'_\beta : U'_\beta \to V'_\beta\}$. A continuous map $f : X \to Y$ is called **holomorphic** if for all α, β, $\varphi'_\beta \circ f \circ \varphi_\alpha^{-1} : \varphi_\alpha(f^{-1}(U'_\beta) \cap U_\alpha) \to V'_\beta$ is holomorphic.

Theorem 2.2. *Let X, Y be Riemann surfaces, and let $f_j \in \mathrm{Hol}(X,Y)$, $j = 1,2$. Assume that there exists $A \subseteq X$ with a limit point $a \in X$ such that $f_1 = f_2$ on A. Then $f_1 \equiv f_2$.*

Proof. (Sketch) Use the connectedness of the Riemann surfaces to transplant the corresponding result from complex analysis. ☐

Proposition 2.1 (local normal form for $f \in \mathrm{Hol}(X,Y)$)**.** *Let X, Y be Riemann surfaces, and let $f_j \in \mathrm{Hol}(X,Y)$ be non-constant. Let $a \in X$. Then there exist complex charts $\varphi : U \to V$ on X with $a \in U$, $\varphi(a) = 0$ and $\psi : U' \to V$; on Y with $f(a) \in U'$, $\psi(f(a)) = 0$, $U \subseteq f^{-1}(U')$ such that the holomorphic function*

$$F = \psi \circ f \circ \varphi^{-1} : V \to V'$$

is of the form $F(z) = z^k$ for some $k \in \mathbb{N}^+$.

Remark 2.1. The integer k is independent is independent of the charts.

Proof. Take any charts φ, ψ centered at $a, f(a)$. Then $\tilde{F}(z) = (\psi \circ f \circ \varphi^{-1})(z) \in \mathrm{Hol}(\mathrm{neigh}(0,\mathbb{C}))$, and $\tilde{F}(0) = 0$. So $\tilde{F}(z) = z^k g(z)$, where g is holomorphic and non-vanishing. In a simply connected neighborhood of 0, there exists a holomorphic function $h \neq 0$ such that $g = h^k$. The map $\kappa(z) = zh(z)$ is a holomorphic diffeomorphism from $\mathrm{neigh}(0,\mathbb{C}) \to \mathrm{neigh}(0,\mathbb{C})$ by the inverse function theorem. Replace φ by $\kappa \circ \varphi$, we get $[\psi \circ f \circ (\kappa \circ \varphi)^{-1}](z) = z^k$. ☐

We will discuss the integer k more next time.

3 Open Mapping, Maximum Principle, Covering Spaces, and Lifts

3.1 The open mapping and the maximum principle

Last time, we showed a local normal form for holomorphic functions:

Proposition 3.1 (local normal form for $f \in \mathrm{Hol}(X, Y)$)**.** *Let X, Y be Riemann surfaces, and let $f_j \in \mathrm{Hol}(X, Y)$ be non-constant. Let $a \in X$. Then there exist complex charts $\varphi : U \to V$ on X with $a \in U$, $\varphi(a) = 0$ and $\psi : U' \to V$; on Y with $f(a) \in U'$, $\psi(f(a)) = 0$, $U \subseteq f^{-1}(U')$ such that the holomorphic function*

$$F = \psi \circ f \circ \varphi^{-1} : V \to V'$$

is of the form $F(z) = z^k$ for some $k \in \mathbb{N}^+$. The integer k is independent of the choice of charts.

Definition 3.1. The integer k is sometimes called the **multiplicity** of f at a. If $k = k(a) > 1$, then a is called a **ramification point**.

Corollary 3.1. *$f \in \mathrm{Hol}(X, Y)$ has no ramification points if and only if f is a local homeomorphism.*

Proof. For any $x \in X$, there is a neighborhood $U \subseteq X$ such that $f : U \to f(U)$ is a homeomorphism. $\square$

Corollary 3.2 (open mapping theorem)**.** *Let $f \in \mathrm{Hol}(X, Y)$ be non-constant. Then f is open.*

Corollary 3.3 (maximum principle)**.** *Let $f \in \mathrm{Hol}(X, \mathbb{C})$ be non-constant. Then $x \mapsto |f(x)|$ does not attain its maximum.*

Proof. If $\sup_{x \in X} |f(x)| = |f(a)|$ for some a, then $f(X) \subseteq \{|z| \leq |f(a)|\}$. $f(X)$ is open, so $f(X) \subseteq \{|z| > f(a)\}$. $\square$

Remark 3.1. In particular, every holomorphic function on a compact Riemann surface is constant.

3.2 Covering spaces and lifts of mappings

Proposition 3.2. *Let X be a Riemann surface, and let Y be a Hausdorff space with a local homeomorphism $p : Y \to X$. There exists a unique complex structure on Y such that $p : Y \to X$ is holomorphic.*

Proof. Existence: Let $\varphi : U \to V$ be a chart on X such that $p : p^{-1}(U) \to U$ is a homeomorphism. Then $\varphi \circ p : p^{-1}(U) \to V$ is a complex chart on Y. These charts define an atlas. Then p is holomorphic. $\square$

Let X, Y, Z be Hausdorff spaces, let $p : Y \to X$ be a local homeomorphism, and let $f : Z \to X$ be continuous. We want a lift $g : Z \to Y$ of f such that $p \circ g = f$.

$$\begin{tikzcd} & Y \arrow[d, "p"] \\ Z \arrow[ur, dashed, "g"] \arrow[r, "f"'] & X \end{tikzcd}$$

Proposition 3.3 (uniqueness of lifts)**.** *Assume that Z is connected. If g_1, g_2 are lifts of f with $g_1(z_0) = g_2(z_2)$, then $g_1 = g_2$.*

Proof. Let $A = \{z \in Z : g_1(z) = g_2(z)\}$ be closed, and let $z_0 \in A$. A is open: Let $z \in A$, $y \in g_1(z)$. Then there exists a neighborhood V of y such that $p : V \to p(V)$ is a homeomorphism. Let W be a neighborhood of z such that $g_1(w) \subseteq V$, $j = 1, 2$. When $z' \in W$, $p(g_1(z')) = p(g_2(z'))$; p is injective, so $g_1 = g_2$ on W. $\square$

Remark 3.2. Assume that X, Y, Z are Riemann surfaces with both p and f holomorphic. Let $\tilde{f} : Z \to Y$ be a lift of f. Then $\tilde{f}$ is holomorphic: $p \circ \tilde{=} f$, where p is a local biholomorphism, so we can locally invert it to get holomorphy of $\tilde{f}$.

Definition 3.2. Let X, Y be topological spaces. A continuous map $p : Y \to X$ is a **covering map** if for all $x \in X$, there is a neighborhood $U \subseteq X$ such that $p^{-1}(U)$ is of the form $p^{-1}(U) = \bigcup_{k \in K} V_k$, where the V_k are open, disjoint, and $p|_{V_k} : V_k \to U$ is a homeomorphism for all k. We say that U is **evenly covered** by p.

Example 3.1. The function $\mathbb{C} \to \mathbb{C} \setminus \{0\}$ given by $z \mapsto e^z$ is a covering map.

Example 3.2. Let Λ be a lattice in $\mathbb{C}$. The projection map $\mathbb{C} \to \mathbb{C}/\Lambda$ is a covering map.

Theorem 3.1. *Let $p : Y \to X$ be a covering map, and let $\gamma : [0,1] \to X$ be a curve (continuous map) in X. Then for any $y \in p^{-1}(\gamma(0))$, there is a unique lift $\tilde{\gamma}$ of γ with $\tilde{\gamma}(0) = y$.*

$$\begin{tikzcd} & Y \arrow[d, "p"] \\ {[0,1]} \arrow[ur, dashed, "\tilde{\gamma}"] \arrow[r, "\gamma"'] & X \end{tikzcd}$$

Proof. Consider the open cover of $[0,1]$ by sets of the form $\gamma^{-1}(U)$, where $U \subseteq X$ is evenly covered. There exists a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ and open sets $U_k \subseteq X$, $1 \leq k \leq n$ evenly covered by p such that $\gamma([t_{k-1}, t_k]) \subseteq U_k$ for all k (use the existence of a Lebesgue number of the cover). Arguing inductively, assume that we have already constructed a lift $\tilde{\gamma}$ of $[0, t_{k-1}]$, where $k \geq 1$. We have that $p \circ \tilde{\gamma} = \gamma$ on $[0, t_{k-1}]$. In particular, $\tilde{\gamma}(t_{k-1}) \in p^{-1}(U_k) = \bigcup_j V_{k_j}$. So $\tilde{\gamma}(t_{k-i}) \in V_{k_j}$ for some j. We set $\tilde{\gamma}(t) = (p|_{V_{k_j}})^{-1} \circ (\gamma(t))$ for $t_{j-1} \leq t \leq t_k$,s thus lifting $\tilde{\gamma}$ defined on $[0, t_k]$. The uniqueness follows. □

Next time, we will show the existence of universal covering spaces that are simply connected. Eventually, we will show that there are only three such simply connected Riemann surfaces.

4 Lifting of Homotopic Curves and Existence of Lifts

4.1 Lifting of homotopic curves

Last time we introduced the idea of a covering map $p : Y \to X$. It has the following path lifting property:

$$\begin{tikzcd} & Y \arrow[d, "p"] \\ {[0,1]} \arrow[ur, dashed, "\tilde{\gamma}"] \arrow[r, "\gamma"'] & X \end{tikzcd}$$

Remark 4.1. If X is path-connected (ok for Riemann surfaces), then $p : Y \to X$ is surjective: Let $x_0, x_1 \in X$, and let γ be a path joining x_0, x_1. Then for any $y \in p^{-1}(x_0)$, there eis a unique lift $\tilde{\gamma}[0,1] \to Y$ such that $\tilde{\gamma}(0) = y$ and $\tilde{\gamma}(1) \in p^1(x_1)$. This gives rise to a bijection $p^{-1}(x_0) \to p^{-1}(x_1)$. Moreover, the cardinality of $p^{-1}(x)$ is constant.

Theorem 4.1 (lifting of homotopy curves[1])**.** *Let X, Y be Hausdorff, and let $p : Y \to X$ be a local homeomorphism. Let $a, b \in X$, and let $\gamma_0, \gamma_1 : [0,1] \to X$ be paths joining a to b that are homotopic. There exists a continuous deformation $H(t,s) : [0,1] \times [0,1] \to X$ such that $H(t,0) = \gamma_0(t)$, $H(t,1) = \gamma_1(t)$, $H(0,s) = a$, and $H(1,s) = b$.*

Let $\gamma_s(t) = H(t,s)$. Let $a_1 \in p^{-1}(a)$, and assume that each γ_s has a lift $\tilde{\gamma}_s$ to y such that $\tilde{\gamma}_s(0) = a_1$. Then $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$ are homotopic and have the same endpoint.[2]

Proof. Set $\tilde{H}(t,s) = \tilde{\gamma}_s(t)$ for $0 \leq t, s \leq 1$. Let us show first that $\tilde{H}$ is continuous. We claim that there exists some $\varepsilon_0 > 0$ such that $\tilde{H}(t,s)$ is continuous on $[0,\varepsilon_0] \times [0,1]$. We have $H(\{0\} \times [0,1]) = \{a\}$. Let $V \subseteq Y, U \subseteq X$ be neighborhoods of a_1 and a such that $p|_V : V \to U$ is a homeomorphism. By compactness of $[0,1]$ and continuity of H, there exists $\varepsilon_0 > 0$ such that $H([0,\varepsilon_0] \times [0,1]) \subseteq U$. Let $\varphi = (p|_V)^{-1} : U \to V$. The curve $[0,\varepsilon_0] \ni t \mapsto \varphi(\gamma_s(t))$ is a lift of γ_s on $[0,\varepsilon_0]$, $0 \leq s \leq 1$, and by the uniqueness of lifts, $\varphi(\gamma_s(t)) = \tilde{\gamma}_s(t) = \tilde{H}(t,s)$ on $0 \leq t \leq \varepsilon_0$. We get the claim.

We now claim that $\tilde{H}$ is continuous on $[0,1] \times [0,1]$. Assume that the claim fails, and let (t_0, σ) be a point of discontinuity of $\tilde{H}$. Let $\tau = \inf\{t : \tilde{H} \text{ is not continuous at } (t,\sigma)\}$. Then $0 < \varepsilon \leq \tau$. Let $x = H(\tau,\sigma)$ and $y = \tilde{(}\tau,\sigma)$; that is, $t = \tilde{\gamma}_\sigma(t)$, so $y \in p^{-1}(x)$. Let V, U be neighborhoods of y and x such that $p|_V : V \to U$ is a homeomorphism, and let $\varphi = (p|_V)^{-1}$. By continuity of H, there exists $\varepsilon > 0$ such that $H(I_\varepsilon(\tau), I_\varepsilon(\sigma)) \subseteq U$, where $I_\varepsilon(\tau)$ is a neighborhood of τ and $I_\varepsilon(\sigma)$ is a neighborhood of σ. In particular, $\gamma_\sigma(I_\varepsilon(\tau)) \subseteq U$. We can also assume that $\tilde{\gamma}_\sigma(I_\varepsilon(\tau)) \subseteq V$. We get $\tilde{\gamma}_\sigma(t) = \varphi(\gamma_\sigma(t))$ for $t \in I_\varepsilon(\tau)$. Let $t_1 \in I_\varepsilon(\tau)$ with $t_1 < \tau$. Then $\tilde{H}$ is continuous at (t_1, σ), so there is a neighborhood $I_\delta(\sigma)$ of σ with $\delta \leq \varepsilon$ such that $\tilde{H}(t_1, s) \in V$ for $s \in I_\delta(\sigma)$. Now $t \mapsto \tilde{\gamma}_s(t)$ and $t \mapsto \varphi(\gamma_s(t))$ for $t \in I_\varepsilon(\tau)$ are both lifts of $\gamma_s(t)$, and by the uniqueness of lifts, $\tilde{\gamma}_s(t) = \varphi(\gamma_s(t))$. In

[1]This theorem is sometimes called the abstract monodromy theorem.

[2]Professor Hitrik says "some theorems may not be meant to be discussed in public." After seeing the proof of this, you may agree.

particular, $\tilde{H}$ is continuous in a neighborhood of (τ, σ), which contradicts the definition of τ. We get that $\tilde{H}$ is continuous on $[0,1] \times [0,1]$.

We also need to check that $s \mapsto \gamma_s(1)$ is constant. This is continuous and lifts the constant path $s \mapsto b$. By the uniqueness of lifts, $\tilde{\gamma}_s(1) = \tilde{\gamma}_0(1) \in p^{-1}(b)$. $\square$

4.2 Existence of lifts

Theorem 4.2 (existence of lifts)**.** *Let X, Y be Hausdorff spaces, and let $p : Y \to X$ be a covering map. Let Z be a Riemann surface which is simply connected, and let $f : Z \to X$ be continuous. For any $x_0 \in Z$ and $y_0 \in Y$ such that $f(z_0) = p(y_0)$, there is a unique lift $\tilde{f} : Z \to Y$ such that $\tilde{f}(z_0) = y_0$.*

$$\begin{tikzcd} & Y \arrow[d, "p"] \\ Z \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

We will prove this next time. First, here are examples.

Example 4.1. Let $Y = \mathbb{C}$ and $X = \mathbb{C} \setminus \{0\}$. Then $p(z) = e^z$ is a covering map. If $f \in \mathrm{Hol}(Z)$ is nonvanishing, then there exists a holomorphic lift $\tilde{f}$ such that $e^{\tilde{f}} = f$.

Example 4.2 (Picard's little theorem)**.** Let $f \in \mathrm{Hol}(\mathbb{C})$ and $0, 1 \notin f(\mathbb{C})$. Then $f : \mathbb{C} \setminus \{0,1\}$. We shall show that the disc D covers $\mathbb{C} \setminus \{0,1\}$:

$$\begin{tikzcd} & D \arrow[d, "p"] \\ \mathbb{C} \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & \mathbb{C} \setminus \{0,1\} \end{tikzcd}$$

Then $\tilde{f} : \mathbb{C} \to D$ is constant, as it is bounded and entire. So f is constant.

5 Existence of Lifts, Germs, and Analytic Continuation

5.1 Existence of lifts

Theorem 5.1 (existence of lifts)**.** *Let X, Y be Hausdorff spaces, and let $p : Y \to X$ be a covering map. Let Z be a Riemann surface which is simply connected, and let $f : Z \to X$ be continuous. For any $z_0 \in Z$ and $y_0 \in Y$ such that $f(z_0) = p(y_0)$, there is a unique lift $\tilde{f} : Z \to Y$ such that $\tilde{f}(z_0) = y_0$.*

$$\begin{tikzcd} & Y \arrow[d, "p"] \\ Z \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

Proof. Let $z \in Z$, and let γ be a path in Z connecting z_0 to z. Then $\alpha : f \circ \gamma$ is a path in X from $f(z_0)$ to $f(z)$. Let $\tilde{\alpha}$ be the unique lift of α starting with $\tilde{\alpha}(0) = y$. Define $\tilde{f}(z) = \tilde{\alpha}(1)$. This does not depend on the choice of γ: this follows as Z is simply connected, using the homotopy lifting lemma. Now $p \circ \tilde{f} = f$, so $\tilde{f}$ is a lift of f.

We need to check the continuity of $\tilde{f}$. Let $z \in Z$, let $y = \tilde{f}(z)$, and let V, U be neighborhoods of $y, p(y)$, respectively such that $p|_V : V \to U$ is a homeomorphism; $y \in V$ and $f(z) \in U$. f is continuous, so there exists a neighborhood W of z which is path-connected such that $f(W) \subseteq U$. We claim that $\tilde{f}(W) \subseteq W$; this will show the continuity of $\tilde{f}$. Let $z' \in W$, and let γ' be a curve in W from z to z'. Let γ and $\alpha = f \circ \gamma$ be as before. Then $\alpha' = f \circ \gamma' \in U$, so $\tilde{\alpha}'$ sending $t \mapsto (p|_V)(\alpha'(t))$ is a lift of α' starting at y. The product curve

$$\tilde{\alpha} * \tilde{\alpha}'(t) = \begin{cases} \tilde{\alpha}(2t) & 0 \leq t \leq 1/2 \\ \tilde{\alpha}'(2t-1) & 1/2 < t \leq 1 \end{cases}$$

is a lift of $\alpha * \alpha' = f(\gamma * \gamma')$. The curve $\gamma * \gamma'$ starts at z_0 and ends at z'. By definition, $\tilde{f}(z') = \tilde{\alpha} * \tilde{\alpha}'(1) = \tilde{\alpha}'(1) \in V$, where V is a small neighborhood of $y = \tilde{f}(z)$. □

5.2 Germs of holomorphic functions

Definition 5.1. Let X be a Riemann surface, and let $a \in X$. If f, g are holomorphic near a, we say that f and g are **equivalent** if there exists a neighborhood W of a such that $f|_W = g|_W$. The equivalence class of f, denoted by f_a is called the **germ** of f at a. We let O_a denote the **space of holomorphic germs** at a.

Remark 5.1. O_a is an algebra (in particular a ring) with no zero divisors.

Let $O_X = \coprod_{a \in X} O_a$. Equip O_X with the following topology. Let $\omega \subseteq X$ be open, and let $f \in \mathrm{Hol}(\omega)$. Set $N(f, \omega) = \{f_x \in O_x : x \in \omega\} \subseteq O_X$. The class of set $N(f, \omega)$ is a base for a topology on O_X, where the open sets are all unions of sets of the form $N(f, \omega)$. If $f' \in \mathrm{Hol}(\omega')$, $f'' \in \mathrm{Hol}(\omega'')$, then $N(f', \omega') \cap N(f'', \omega) = N(f', \omega) = N(f'', \omega)$, where $\omega = \{x \in \omega' \cap \omega'' : f'_x = f''_x\}$ is open.

Definition 5.2. The topological space O_X is called the **sheaf of germs** of holomorphic functions on X.

We have the natural map $p : O_X \to X$ sending $f_a \mapsto a$.

Proposition 5.1. *p is a local homeomorphism.*

Proof. Let $f_a \in O_X$, and let (f, ω) be a representative of f_a. Then $p : N(f, \omega) \to \omega$ is a homeomorphism. $\square$

Remark 5.2. This means that we can given O_X the structure of a Riemann surface. However, this is not a covering map.

Proposition 5.2. *The topological space O_X is Hausdorff.*

Proof. Let $f_a, g_b \in O_X$ with $f_a \neq g_b$. If $a \neq b$, there exist representatives $(f, \omega_a), (g, \omega_b)$ with $\omega_a \cap \omega_b = \varnothing$ such that $N(f, \omega_a) \cap N(g, \omega_b) = \varnothing$. If $a = b$ and $f_a \neq g_a$, then there exists a connected neighborhood ω of a and representatives $f(f, \omega), (g, \omega)$ such that $N(f, \omega) \cap N(g, \omega) = \varnothing$ by analytic continuation. $\square$

5.3 Analytic continuation

Definition 5.3. Let $a \in X$, $f_a \in O_a$, and let γ be a curve in X starting at a. The **analytic continuation** of f_a along γ is a lift $\tilde{\gamma} : [0, 1] \to O_X$ of γ such that $\tilde{\gamma}(0) = f_a$.

$$\begin{tikzcd} & O_X \arrow[d, "p"] \\ Z \arrow[ur, dashed, "\tilde{\gamma}"] \arrow[r, "\gamma"'] & X \end{tikzcd}$$

We write $\tilde{\gamma}(t) = f_{\gamma(t)} \in O_{\gamma(t)}$.

Remark 5.3. The analytic continuation, if it exists, is unique (uniqueness of lifts).

Example 5.1. It is not always possible to find an analytic continuation. Let $\gamma(t) = t$ for $0 \leq t \leq 1$, and let $f(z) = 1/(1 - z)$ near 0. Then f cannot be analytically continued along the curve γ.

6 The Monodromy Theorem and Application to Linear ODE

6.1 The monodromy theorem

Last time, we introduced the notion of analytic continuation. If $a \in X$, and $f_a \in O_a$, then an analytic continuation along some curve $\gamma : [0,1] \to X$ is a lift $\tilde{\gamma}$ to the sheaf of germs such that for all $t \in [0,1]$, $\tilde{\gamma}(t) = f_{\gamma(t)} \in O_{\gamma(t)}$, and $t \mapsto f_{\gamma(t)}$ is continuous.

$$\begin{tikzcd} & O_X \arrow[d, "p"] \\ Z \arrow[r, "\gamma"'] \arrow[ur, dashed, "\tilde{\gamma}"] & X \end{tikzcd}$$

That is, for all $t_0 \in [0,1]$, there is a neighborhood $I_{t_0} \subseteq [0,1]$ of t_0 and an open set $\omega \subseteq X$ such that $\gamma(I_{t_0}) \subseteq \omega$, and $\tilde{f} \in \mathrm{Hol}(\omega)$: $\tilde{f}_{\gamma(t)} = f_{\gamma(t)}$ for all $t \in I_{t_0}$.

Theorem 6.1 (monodromy theorem)**.** *Let X be a Riemann surface, let $a, b \in X$, and let γ_0, γ_1 be homotopic curves from a to b. Let $f_a \in O_a$. Let $H(t,s)$ be a homotopy between γ_0 and γ_1, and assume that f_a has an analytic continuation $\tilde{\gamma}_s$ along $\gamma_s(t) = H(t,s)$ for all s. Then $s \mapsto \gamma_s(1) \in O_b$ are equal for all s. In particular, $\tilde{\gamma}_0(1) = \tilde{\gamma}_1(1)$.*

Proof. Apply the homotopy lifting theorem to the local homeomorphism $p : O_X \to X$. $\square$

Corollary 6.1. *Let X be a simply connected Riemann surface, and let $a \in X$. Let $f_a \in O_a$. be a holomorphic germ which can be continued along any curve starting at a. Then there exists a unique globally defined holomorphic function $F \in \mathrm{Hol}(X)$ such that $F_a = f_a$ in O_a.*

Proof. When $x \in X$, let γ be a path from a to x, and let $f_x \in O_x$ be the analytic continuation of f_a along γ (f_x is independent of the choice of γ). Define $F(x) = f_x(x)$. $\square$

6.2 Linear ODE in the complex domain

Here is the historical origin of the idea of monodromy. This will be a good example of the applications of our theory.

Proposition 6.1. *Let $\Omega \subseteq \mathbb{C}$ be open, and let $A \in \mathrm{Hol}(\Omega, \mathrm{Mat}_{n\times n}(\mathbb{C}))$. Let Ω be simply connected. Then for all $z_0 \in \Omega$ and $x_0 \in \mathbb{C}^n$, then Cauchy problem*

$$x'(z) = A(z)x(z), \qquad x(z_0) = x_0$$

has a unique solution $x(z) \in \mathrm{Hol}(\Omega, \mathbb{C}^n)$

Proof. (idea) Write

$$x(z) = x_0 + \int_{\gamma_{z_0,z}} A(\zeta)x(\zeta)\,d\zeta,$$

and solve the integral equation by Picard's iterations. $\square$

Assume now that $\Omega = \{0 < |z| < 1\}$ is not simply connected. We have the covering map $e^\zeta : \{\mathrm{Re}(\zeta) < 0\} \to \{0 < |z| < 1\}$, and we can lift the ODE to $\{\mathrm{Re}(\zeta) < 0\}$. If we let $y(\zeta) = z(e^\zeta)$, then

$$y'(\zeta) = \underbrace{e^\zeta A(e^\zeta)}_{2\pi i\text{-periodic}} y(\zeta).$$

We argue more directly: Let $\omega \subseteq \Omega$ be a small, simply connected neighborhood of $z_0 \in \{0 < |z| < 1\}$, and let $V(\omega) = \{x(z) \in \mathrm{Hol}(\omega, \mathbb{C}^n) : x'(z) = A(z)z(z) \text{ in } \omega\}$. This is an n-dimensional vector space. We can continue elements of $V(\omega)$ analytically: let $\Gamma_1 = \{z \in \Omega\text{''}\alpha < \arg(z) < \beta\}$ with $\alpha < 0$, $\beta > \pi$, and $\Gamma_1 \supseteq \omega$. Then $V(\Gamma_1)$ is the set of solutions to the ODE in Γ_1. We have the extension map $E : V(\omega) \to V(\Gamma_1)$. We then restrict to a domain ω' on the other side of the disc, extend to another sector Γ_2, and restrict to ω. We get a linear bijective map $S : V(\omega) \to V(\omega)$ called the **monodromy map** of this ODE.

Let $x_1, \dots, x_n$ be a basis for $V(\omega)$, and let $F(z) = \begin{bmatrix} x_1(z) & \cdots & x_n(z) \end{bmatrix}$ be the fundamental matrix with columns x_i. Write

$$Sx_j(z) = \sum_k S_{k,j} x_k(z).$$

If we denote $x_1(ze^{2\pi i}) = Sx_j(z)$, we get

$$F(ze^{2\pi i}) = F(z)A$$

for $z \in \omega$. We claim that there exists a matrix C such that $F(z) = Q(z)z^C$ in ω, where $Q(z) \in \mathrm{Hol}(0 < |z| < 1)$ and $z^C = e^{C\log(z)}$. To get the claim, we write $S = e^2\pi iC$ and check that $Q(z)$ satisfies $Q(ze^{2\pi i}) = Q(z)$.

6.3 Analytic continuation to larger Riemann surfaces

Let X be a Riemann surface, and let $\varphi \in O_a$ for some $a \in X$. We would like to construct a new Riemann surface which arises by analytic continuation of φ.

Definition 6.1. An **analytic continuation** of φ is given by (Y, p, f, b), where Y is a Riemann surface, $p : Y \to X$ is holomorphic with no ramification points, $f \in \mathrm{Hol}(Y)$, $b \in p^{-1}(a)$, and $f_b = p^*(\varphi)$. Here, p^* is the pullback map $p^*(\varphi) = \varphi \circ p$.

7 Maximal Analytic Continuation and Analytic Functionals

7.1 Maximal analytic continuation

Let X, Y be Riemann surfaces, and let $p : Y \to X$ be holomorphic with no ramification points. Then p is a local biolomorphism, and the pullback map $p^* : O_{X,p(y)} \to O_y$ sending $f \mapsto f \circ p$ is an isomorphism with inverse p_*. Let $\varphi \in O_{X,a}$ for some $a \in X$.

Definition 7.1. An **analytic continuation** of φ is given by (Y, p, f, b), where $p : Y \to X$ is holomorphic and unramified, $f \in \mathrm{Hol}(Y)$, $b \in p^{-1}(a)$, and $p_*(f_b) = \varphi$.

Definition 7.2. An analytic continuation is **maximal** if the following property holds: if (Z, q, g, c) is another continuation of φ, then there exists a holomorphic map $F : Z \to Y$ which is fiber preserving ($p \circ F = q$) such that $F(c) = b$ and $F^* f = g$.

Theorem 7.1. *Let X be a Riemann surface, $\varphi \in O_{X,a}$. Then there exists a maximal analytic continuation (Y, p, f, b) of φ.*

Remark 7.1. One can show that this is unique up to holomorphic diffeomorphism, but we will not do that here.

Lemma 7.1. *Let (Y, p, f, b) be an analytic continuation of φ. Let $\gamma : [0, 1] \to Y$ be a path in Y from b to $y \in Y$. Then the germ $\psi = p_*(f_y) \in O_{X,p(y)}$ is an analytic continuation of φ along the path $p \circ \gamma$.*

Proof. Set $\varphi_t = p_*(f_{\gamma(t)}) \in O_{x,p(\gamma(t))}$ for all $0 \leq t \leq 1$. Then $\varphi_0 = \varphi$, and $\varphi_q = \psi$. We need to check that $[0, 1] \to O_X$ sending $t \mapsto \varphi_t$ is continuous. Let $t_0 \in [0, 1]$. Then there exist neighborhoods $V \subseteq Y$ of $\gamma(t_0)$ and $U \subseteq X$ of $p(\gamma(t_0))$ such that $p|_V : V|toU$ is a holomorphic bijection. Let $g = f \circ ((p|_V)^{-1}) \in \mathrm{Hol}(U)$. Then $p_*(f_z) = g_{p(z)}$ for all $z \in V$. We can find a neighborhood I_{t_0} of t_0 such that $\gamma(I_{t_0}) \subseteq V$. Then for every $t \in I_{t_0}$, $\varphi_t = g_{p(\gamma(t))}$. Thus, ψ is an analytic continuation of φ along $p \circ \gamma$. □

Now let's prove the theorem.

Proof. Let Y be the connected component in O_X containing φ. Then $Y \subseteq O_X$ is open (since O_X is locally connected), and the map $p = p|_Y$ is a local homeomorphism $Y \to X$. There exists a unique complex structure on Y such that $p : Y \to X$ is holomorphic. Let $\zeta \in Y$. Then ζ is a germ of a holomorphic function on X at $p(\zeta)$. Define $f(\zeta) = \zeta(p(\zeta))$. Then $f \in \mathrm{Hol}(Y)$, and if $b = \varphi$, then $b \in p^{-1}(a)$ and $p_*(f_b) = \varphi$.

Let us check the maximality of (Y, p, f, b). Let (Z, q, g, c) be an analytic continuation of φ. Let $z \in Z$ and $z = q(z)$. The germ $q_*(g_z) \in O_{X,x}$ arises by analytic continuation of φ along a curve from a to x in X. Thus, there exists a unique $\psi \in Y$ such that $q_*(g_z) = \psi$. We get a map $F : Z \to Y$ sending $z \mapsto \psi$, and it follows that (Y, p, f, b) is maximal. □

7.2 Analytic functionals and the Fourier-Laplace transform

Definition 7.3. We say that a linear map $\mu : \mathrm{Hol}(\mathbb{C}) \to \mathbb{C}$ is an **analytic functional** if it is continuous in the following sense: there exist a compact $K \subseteq \mathbb{C}$ and constant $C > 0$ such that $|\mu(f)| \leq C \sup_K |f|$ for all $f \in \mathrm{Hol}(\mathbb{C})$.

Remark 7.2. By the Hahn-Banach theorem, μ can be extended to a linear continuous functional on $C(K)$. Then there exists a measure ν on K such that $\mu(f) = \int_K f(z)\,\nu(z)$ for $f \in \mathrm{Hol}(\mathbb{C})$.

Example 7.1. Let $\gamma : [0,1] \to \mathbb{C}$ be a C^1 path, and define the functional $\mu(f) = \int_\gamma f(z)\,dz = \int_0^1 f(\gamma(t))\gamma'(t)\,dt$. μ does not change if γ if replaced by a homotopic path. So the representing measure need not be unique.

Example 7.2. Let $\mu(f) = f^{(j)}(0)$ for $j \geq 0$ is an analytic functional.

Definition 7.4. A compact set $K \subseteq \mathbb{C}$ is called a **carrier** for the analytic functional μ if for every open neighborhood ω of K, there is a constant C_ω such that $|\mu(f)| \leq C_\omega \sup_\omega |f|$ for $f \in \mathrm{Hol}(\mathbb{C})$.

Remark 7.3. The first example shows that carriers need not be unique, either.

Definition 7.5. The **Fourier-Lapclace transform** $\widehat{\mu}$ of μ is defined by

$$\widehat{\mu}(\zeta) = \mu_z(e^{z\zeta}), \qquad \zeta \in \mathbb{C}.$$

We have that $\widehat{\mu}$ is entire (by its description as integration of this function against a measure).

Proposition 7.1. *The map $\mu \mapsto \widehat{\mu}$ is injective.*

Proof. If $\widehat{\mu}(\zeta) = 0$ for all ζ, then $0 = \partial_\zeta^j \widehat{\mu}|_{\zeta=0} = \mu(z^j)$ for all j. In particular, for any polynomial p, $\mu(p) = 0$. Polynomials are dense in $\mathrm{Hol}(\mathbb{C})$, so $\mu(f) = 0$ for all $f \in \mathrm{Hol}(\mathbb{C})$. That is, $\mu(f) = 0$. $\square$

8 Inversion of the Fourier-Laplace Transform

8.1 Bounds on analytic functionals

Last time, we were talking about analytic functionals $\mu : \mathrm{Hol}(\mathbb{C}) \to \mathbb{C}$. We defined the Fourier-Laplace transform $\widehat{\mu}(\zeta) = \mu_z(e^{z\zeta})$, $z \in \mathbb{C}$. Assume that μ is carried by the compact set $K \subseteq \mathbb{C}$: for all neighborhoods ω of K,

$$|\mu(f)| \leq C_\omega \sup_\omega |f|, \qquad f \in \mathrm{Hol}(\mathbb{C}).$$

So there exists a measure ν on $\overline{\omega}$ such that

$$\mu(f) = \int_{\overline{\omega}} f(z)\, d\nu(z).$$

So we get the bound

$$|\widehat{\mu}(\zeta)| \leq \exp(\sup_{z \in \overline{\omega}} \mathrm{Re}(z\zeta)) \int_{\overline{\omega}} |d\nu(z)|.$$

Ir follows that for any $\delta > 0$, there is a constant C_δ such that

$$|\widehat{\mu}(\zeta)| \leq C_\delta \exp(H_K(\zeta) + \delta|\zeta|), \qquad \zeta \in \mathbb{C},$$

where

$$H_K(\zeta) = \sup_{z \in K} \mathrm{Re}(z\zeta)$$

is the **support function** of K. H_K is a convex, positively homogeneous of $\zeta \in \mathbb{C} \cong \mathbb{R}^2$. In particular, $\widehat{\mu}$ is entire of order 1 and of **exponential type**:

$$|\widehat{\mu}(\zeta)| \leq Ce^{a|\zeta|}.$$

Proposition 8.1. *Let K be compact and convex with the support function H_K. Then $K = \{z \in \mathbb{C} : \mathrm{Re}(z\zeta) \leq H_K(\zeta)\ \forall \zeta \in \mathbb{C}\}$.*

Proof. ($\subseteq$): This inclusion is by definition of H_K.

($\supseteq$): Let $z_0 \notin K$. By the geometric Hahn-Banach theorem, there exists a hyperplane separating K and z_0. That is, there exists a real, linear form f on $\mathbb{R}^2$ and $\gamma \in \mathbb{R}$ such that $f(z) < \gamma < f(z_0)$ for any $z \in K$. There is a $\zeta \in \mathbb{C}$ such that $f(z) = \mathrm{Re}(z\zeta)$, so $H_K(\zeta) < \mathrm{Re}(z_0\zeta)$. □

To summarize, if μ is carried by a compact K, then its transform $\mathcal{M}(\zeta) = \widehat{\mu}(\zeta)$ is entire and satisfies: for all $\delta > 0$, there exists a C_δ such that

$$|\mathcal{M}(\zeta)| \leq C_\delta \exp(H_K(\zeta) + \delta|\zeta|).$$

8.2 Inversion of the Fourier-Laplace transform

Theorem 8.1 (Polya, Ehrenpreis, Martineau[3]). *Let $K \subseteq \mathbb{C}$ be compact and convex, and let $\mathcal{M} \in \mathrm{Hol}(\mathbb{C})$ be such that*

$$|\mathcal{M}(\zeta)| \le C_\delta \exp(H_K(\zeta) + \delta|\zeta|).$$

Then there exists a unique analytic functional μ such that $\widehat{\mu} = \mathcal{M}$ and μ is carried by K.

Proof. Idea: Construct the analytic functional μ using the **Borel transform** of $\mathcal{M}$. In particular, the estimate on $\mathcal{M}$ gives

$$|\mathcal{M}(\zeta)| \le C_1 e^{C|\zeta|}$$

for some C_1, C. When $R > 0$, we have

$$\frac{\mathcal{M}^{(j)}(0)}{j!} = \frac{1}{2\pi i} \int_{|\zeta|=R} \frac{\mathcal{M}(\zeta)}{\zeta^{j+1}}\, d\zeta,$$

which gives

$$|M^{(j)}| \le j! C_1 e^{CR} R^{-1}.$$

The optimal choice of R is given by $R = j/C$. So we get

$$|\mathcal{M}^{(j)}(0)| \le j! C_1 e^j \left(\frac{C}{j}\right)^j \le C_1 (Ce)^j, \qquad j = 0, 1, 2, \ldots.$$

Define

$$B(\zeta) = \sum_{j=0}^{\infty} \zeta^{-j-1} \mathcal{M}^{(j)}(0).$$

Then $B \in \mathrm{Hol}(\hat{\mathbb{C}} \setminus \{|\zeta| \le Ce\})$, and $B(\infty) = 0$. Then function B is called the Borel transform of $\mathcal{M}$.

Let $\chi \in C_0^\infty(\mathbb{C})$ be such that $\chi = 1$ on a large disc, and define

$$\mu(f) = -\frac{1}{\pi} \iint \frac{\partial \chi}{\partial \bar{\zeta}}(\zeta) f(\zeta) B(\zeta)\, d\lambda(\zeta),$$

where λ is Lebesgue measure in $\mathbb{C}$. Then μ is an analytic functional which is independent of the choice of χ. We claim first that $\widehat{\mu} = \mathcal{M}$: compute

$$\widehat{\mu}^{(j)}(0) = \mu(\zeta^j)$$

[3] Polya proved the theorem in complex dimension 1. Ehrenpreis and Martineau generalized it to $\mathbb{C}^n$ for $n > 1$.

$$= -\frac{1}{\pi}\iint \frac{\partial \chi}{\partial \overline{\zeta}}(\zeta)\zeta^j B(\zeta)\, d\lambda(\zeta)$$
$$= \sum_{k=0}^{\infty} -\frac{1}{\pi}\iint \frac{\partial \chi}{\partial \overline{\zeta}}(\zeta)\zeta^j \zeta^{-k-1}\mathcal{M}^{(k)}(0)\, d\lambda(\zeta)$$

When $k = j$, the summand is

$$\mathcal{M}^{(j)}(0)\underbrace{\left(-\frac{1}{\pi}\iint \frac{\partial \chi}{\partial \overline{\zeta}}(\zeta)\frac{1}{\zeta}\, d\lambda(\zeta)\right)}_{=1}$$

by the Cauchy integral formula. When $j \neq k$, it equals

$$\iint \frac{\partial \chi}{\partial \overline{\zeta}}(\zeta)\zeta^{\nu}\, d\lambda(\zeta),$$

where $\nu \neq -1$. We can choose $\chi(\zeta) = \psi(|\zeta|^2)$ (making it radially symmetric to get:

$$\iint \psi'(|\zeta|^2)\zeta^{\nu+1}\, d\lambda(\zeta) = \iint \psi'(|\zeta|^2)r^{\nu+1}e^{i\theta(\nu+1)}r\, dr\, d\theta = 0.$$

We get $\widehat{\mu}^{(j)}(0) = \mathcal{M}^{(j)}(0)$. So $\widehat{\mu} = \mathcal{M}$ as their Taylor expansions agree.

We claim that B can be continued analyrically to $\hat{\mathbb{C}} \setminus K$. We will do this next time. $\square$

9 Polya's Theorem and Universal Covering Spaces

9.1 Polya's theorem (cont.)

Last time, we were proving Polya's theorem. Let's finish the proof.

Theorem 9.1 (Polya, Ehrenpreis, Martineau)**.** *Let $K \subseteq \mathbb{C}$ be compact and convex, and let $\mathcal{M} \in \mathrm{Hol}(\mathbb{C})$ be such that*

$$|\mathcal{M}(\zeta)| \leq C_\delta \exp(H_K(\zeta) + \delta|\zeta|).$$

Then there exists a unique analytic functional μ such that $\widehat{\mu} = \mathcal{M}$ and μ is carried by K.

Proof. Set

$$\mu(f) = -\frac{1}{\pi} \iint \frac{\partial \chi}{\partial \overline{\zeta}}(\zeta) f B \, d\lambda(\zeta),$$

where $\chi \in C_0^\infty(\mathbb{C})$ is 1 on a large disc and B is the Borel transform of $\mathcal{M}$. We claim that B can be extended analytically to $\hat{\mathbb{C}} \setminus K$. First, if the claim holds, μ is carried by K: for any neighborhood ω of K, we can choose $\chi \in C_0^\infty(\omega)$ such that $\chi = 1$ in a neighborhood of K.

Proof of claim: Let $w \in \mathbb{C}$ with $|w| = 1$, and let

$$B_w(\zeta) = \int_0^\infty \mathcal{M}(tw) w e^{-tw\zeta} \, dt.$$

We have

$$|\mathcal{M}(tw)e^{-tw\zeta}| \leq C_\delta \exp(tH_K(w) + \delta t - t \,\mathrm{Re}(w\zeta)).$$

Let $\Pi_w = \{\zeta \in \mathbb{C} : \mathrm{Re}(w\zeta) > H_K(w)\}$. It follows that $B_w \in \mathrm{Hol}(\Pi_w)$. When $\zeta \in \mathbb{C}$ is such that $w\zeta$ is real and $\gg 0$, then we can compute $B_w(\zeta)$ by expanding $\mathcal{M}(tw) = \sum_{j=0}^\infty \mathcal{M}^{(j)}(0)(tw)^j/j!$ as a Taylor series and integrating term by term. In general, if $f \in \mathrm{Hol}(|z| < R)$ and $|f| \leq M$, then Cauchy's estimates give

$$\left| f(z) - \sum_{j=0}^{n-1} \frac{f^{(j)}(0)}{j} z^j \right| \leq \sum_{j=n}^\infty \frac{|f^{(j)}(0)|}{j!} |z|^j \leq M \left(\frac{|z|}{R}\right)^n \frac{1}{1 - |z|/R},$$

so integrating the Taylor series term by term is justified.

We get

$$B_w(\zeta) = \sum_{j=0}^\infty \frac{\mathcal{M}^{(j)}(0)}{j!} w^{j+1} \underbrace{\int_0^\infty t^j e^{-tw\zeta} \, dt}_{=j!(w\zeta)^{-(j+1)}} = B(\zeta)$$

for any w. It follows that for any w_1, w_2, B_{w_1}, B_{w_2} coincide in the region $\Pi_{w_1} \cap \Pi_{w_2}$, for they are both equal to B far away. We get a well-defined holomorphic function on $\bigcup_{|w|=1} \Pi_w$ which analytically continues B. Now

$$\bigcup_{|w|=1} \Pi_w = \{\zeta \in \mathbb{C} : H_k(w) < \operatorname{Re}(w\zeta) \text{ for some } w\} = \mathbb{C} \setminus K,$$

as we checked that $K = \{\zeta : \operatorname{Re}(z\zeta) \leq H_K(z)\, \forall z \in \mathbb{C}\}$. $\square$

Remark 9.1. Let μ be an analytic functional. Then there is a compact set $K \subseteq \mathbb{C}$ and a measure ν on K such that

$$\mu(f) = \int_K f(z)\, d\nu(z).$$

By Cauchy's integral formula,

$$f(z) = -\frac{1}{\pi} \iint \frac{\partial \chi}{\partial \overline{\zeta}}(\zeta) \frac{f(\zeta)}{\zeta - z}\, d\lambda(s), \qquad z \in K,$$

where $\chi \in C_0^\infty$ equals 1 in a neighorhood of K. Then

$$\mu(f) = -\frac{1}{\pi} \iint \frac{\partial \chi}{\partial \overline{\zeta}}(\zeta) \frac{f(\zeta)}{\zeta - z} \varphi(\zeta)\, d\lambda(s),$$

where

$$\varphi(\zeta) = \int_K \frac{1}{\zeta - z}\, d\nu(z) \in \operatorname{Hol}(\mathbb{C} \setminus K),$$

and at ∞,

$$\varphi(\zeta) = \sum \frac{1}{\zeta^{j+1}} \underbrace{\left(\int z^j\, d\nu(z) \right)}_{=\mu(z^j)} = B(\zeta).$$

So it is natural to look for this kind of representation of an analytic functional.

9.2 Universal covering spaces

Theorem 9.2. *Let X be a connected topological manifold. Then there exists a simply connected manifold $\tilde{X}$ and a covering map $p : \tilde{X} \to X$.*

Remark 9.2. If $\tilde{p} : \tilde{X} \to X$ and $\hat{p} : \hat{X} \to X$ are covering maps and $\tilde{X}, \hat{X}$ are simply connected, then there is a homeomorphism $f : \tilde{X} \to \hat{X}$ such that $\hat{p} \circ f = \tilde{p}$.

Proof. Let $x_0 \in C$, and let $\pi(x_0, x)$ be the set of homotopy classes of paths from x_0 to x. Define $\tilde{X} = \{(x, \Gamma) : x \in X, \Gamma \in \pi(x_0, x)\}$. Define the following topology on $\tilde{X}$: Let $(x, \Gamma) \in \tilde{X}$, and let U be a path-connected and simply connected neighborhood of X.

Define $\langle U, \Gamma \rangle = \{(y, \Gamma) : y \in U, \Lambda = [\gamma * \alpha], \Gamma = [\gamma], \alpha \text{ from } x \text{ to } y\}$. Use the sets $\langle U, \Gamma \rangle$ as a base for a topology on $\tilde{X}$.

Let $p : \tilde{X} \to X$ send $(x, \Gamma) \mapsto x$. We claim that p is a covering map. Let $x \in X$, and let U be a path-connected and simply connected neighborhood of x. Then

$$p^{-1}(U) = \bigcup_{p(x,[\sigma])=x} \langle U, [\sigma] \rangle \,,$$

where σ is a path from x_0 to x. If $[\sigma] \neq [\tau]$, then $\langle U, [\sigma] \rangle \neq \langle U, [\tau] \rangle$: if $(y, [\gamma]) \in \langle U, [\sigma] \rangle \cap \langle U, [\tau] \rangle$, then there are paths α, β in U from x to y such that $[\gamma] = [\sigma * \alpha] = [\tau * \beta]$; α and β are homotopic, so $[\sigma] = [\tau]$.

One checks that p is continuous and open. Let us see that $p : \langle U, [\sigma] \rangle \to U$ is bijective:

- surjective: U is path-connected. p is injective:
- injective: Suppose $(y, [\tau]) = p(y, [\gamma])$. Then there are paths α, β from x to y such that $[\tau] = [\sigma * \alpha]$ and $[\gamma] = [\sigma * \beta]$. α and β are homotopic, so $[\tau] = [\gamma]$.

We have checked that $p : \tilde{X} \to X$ is a covering map.

It remains to show that $\tilde{X}$ is simply connected. We will do this next time. $\square$

10 Simply Connectedness of Universal Covering Spaces and Green's Functions

10.1 Simply connectedness of universal covering spaces

Last time, we were proving the existence of universal covering spaces.

Theorem 10.1. *Let X be a connected topological manifold. Then there exists a simply connected manifold $\tilde{X}$ and a covering map $p : \tilde{X} \to X$.*

Proof. Let $\tilde{X} = \{(x, [\sigma]) : \sigma \text{ is a path in } X \text{ from } x_0 \text{ to } x\}$. We have shown that $p : \tilde{X} \to X$ sending $(x, [\sigma]) \mapsto x$ is a covering map. We claim that $\tilde{X}$ is simply connected. When $|sigma$ is a path in X from x_0 to $x \in X$, consider the path in $\tilde{X}$: $\sigma' : [0, 1] \to X$ with $\sigma'(s) = (\sigma(s), [t \mapsto \sigma(ts)]) \in \tilde{X}$. Then $\sigma'(0) = (x_0, [\varepsilon_{x_0}])$ (w here ε_{x_0} is the constant path at x_0), and $\sigma'(1) = (x, [\sigma])$. Moreover, $p \circ \sigma' = \sigma$. So $\tilde{X}$ is path-connected.

Let σ'' be a closed path in $\tilde{X}$ with $\sigma''(0) = \sigma''(1) = (x_0, [\varepsilon_{x_0}])$. Then $\sigma := p \circ \sigma''$ is a closed path in X starting and ending at x_0. The path σ can be lifted to $\tilde{X}$, and by the uniqueness of lifts, σ'' sends $[0, 1] \ni s \mapsto (\sigma(s), [t \mapsto \sigma(st)]) \in \tilde{X}$. Thus, $(x_0, [\varepsilon_{x_0}]) = \sigma''(0) = \sigma''(1) = (x, [\sigma])$, so σ is null-homotopic in X. By the homotopy lifting theorem, σ'' is null-homotopic in $\tilde{X}$. $\square$

10.2 Green's functions in $\mathbb{C}$

We want to prove the uniformization theorem:

Theorem 10.2 (Poincaré, Koebe)**.** *Let X be a simply connected Riemann surface. Then X is complex diffeomorphic to $\hat{\mathbb{C}}$, $\mathbb{C}$, or the unit disc $D \subseteq \mathbb{C}$.*

Here is the starting point of the proof. We will try to construct a Green's function for X. Recall the notion of a Green's function for an open, bounded $\Omega \subseteq \mathbb{C}$ with C^2 boundary.

Definition 10.1. We say that $G(x, y)$ for $x \in \Omega$, $y \in \overline{\Omega}$ is a **Green's function** for Ω if

1. $G(x, y) = \frac{1}{2\pi} \log |x - y| + h_x(y)$, where $h_x \in C^2(\overline{\Omega})$ is harmonic in Ω.
2. $G(x, y) = 0$ for $y \in \partial\Omega$.

Remark 10.1. If G exists, it is unique. The function $y \mapsto G(x, y)$ is subharmonic in Ω. By the maximum principle, $G(x, y) < 0$ for all $(x, y) \in \Omega \times \Omega$.

Assume that $G(x, y)$ exists, and let $u \in C^2(\overline{\Omega})$ with $u|_{\partial\Omega}$. Cut out a small disc around x to get $\Omega_\varepsilon = \{y \in \Omega : |x - y| > \varepsilon\}$. By Green's formula,

$$\int_{\Omega_\varepsilon} (u(y)\Delta_y G(x, y) - G(x, y)\Delta u(y)) = \int_{\partial\Omega_\varepsilon} \left(u(y)\frac{\partial G(x, y)}{\partial n_y} - G(x, y)\frac{\partial u}{\partial n_y} \right) ds(y)$$

$$= \cancelto{0}{\int_{\partial\Omega}} + \int_{S_\varepsilon},$$

where n is the unit outgoing vector, normal to $\partial\Omega_\varepsilon$, and $S_\varepsilon = \{y : |y-x| = \varepsilon\}$. Consider

$$\int_{S_\varepsilon} - \underbrace{G(x,y)}_{=O(\log(1/\varepsilon))} \frac{\partial u}{\partial n_y} \underbrace{ds(y)}_{=O(\varepsilon)} = O(\varepsilon \log(1/\varepsilon)) \xrightarrow{\varepsilon\to 0} 0.$$

Compute also

$$\begin{aligned}
&\int_{S_\varepsilon} u(y)\nabla_y \left(\frac{1}{2\pi}\log|x-y| + h_x(y)\right) \frac{-(y-x)}{|y-x|}\, ds(y) \\
&\qquad = \int_{s_\varepsilon} u(y) \left(\frac{1}{2\pi}\frac{1}{|y-x|}\frac{y-x}{|y-x|}\frac{-(y-x)}{|y-x|} + O(1)\right) ds(y) \\
&\qquad = -\frac{1}{2\pi\varepsilon}\int_{s_\varepsilon} u(y)\, ds(y) + o(1) \\
&\qquad \xrightarrow{\varepsilon\to 0^+} -u(x).
\end{aligned}$$

The left hand side in Green's formula equals

$$-\int_{\Omega_\varepsilon} G(x,y)\Delta u(y)\, dy \to \int_{\Omega} \xrightarrow{\varepsilon\to 0^+} \int_{\Omega} -G(x,y)\Delta u(y)\, dy,$$

where we can use the dominated convergence theorem since $G \in L^1_{\text{loc}}(\Omega)$. We get

$$u(x) = \int_\Omega G(x,y) f(y)\, dy$$

if $f = \Delta u \in C(\overline{\Omega})$. Here, we have used that $u \in C^2(\overline{\Omega})$ and $u|_{\partial\Omega} = 0$.

Assume now that $u \in C_0^2(\mathbb{R}^2)$. Take $\Omega = D(0,R)$ for large $R > 0$, and let $x = 0$. Then

$$u(0) = \int G(0,y)\Delta u(y)\, dy = \int \left(\frac{1}{2\pi}\log|y| + h_0(y)\right)\Delta u(y)\, dy.$$

h_0 is harmonic in $D(0,R)$, so

$$\int h_0 \Delta u(y)\, dy = 0$$

after integrating by parts. So we get that

$$\int E(y)\Delta u(y)\, dy = u(0), \qquad E(y) = \frac{1}{2\pi}\log|y|$$

for all $u \in C_0^2(\mathbb{C})$. When this formula holds, we say that E is a **fundamental solution** of Δ, and we write $\Delta E = \delta_0$, where δ_0 is the Dirac measure at 0: $\delta_0(u) = u(0)$.

To construct $G(x, y)$ for a given Ω, we need to solve

$$\Delta_y h_x(y) = 0$$

in Ω with the boundary condition

$$\left(h_x + \frac{1}{2\pi} \log|x - \cdot|\right)_{\partial\Omega} = 0.$$

This can be solved using Perron's method. We will extend Perron's method to a Riemann surface and construct a Green's function using this method.

11 Weyl's Lemma and Perron's Method

11.1 Weyl's lemma

Last time, we were talking about Green's functions for $\Omega \subseteq \mathbb{C}$:

$$G(x,y) = \frac{1}{2\pi} \log|x-y| + h_x(y), \qquad G(x,y) = 0, y \in \partial\Omega,$$

where h_x is harmonic. If

$$E(x) = \frac{1}{2\pi}\log|x|,$$

then E is a fundamental solution of Δ: for all $\varphi \in C_0^\infty(\mathbb{R}^2)$:

$$\int E\Delta\varphi = \varphi(0).$$

Theorem 11.1 (Weyl's lemma)**.** *Let $\Omega \subseteq \mathbb{C}$ be open, and let $u \in L^1_{\mathrm{loc}}(\Omega)$ be such that*

$$\int u\Delta\varphi \, dx = 0 \qquad \forall \in C_0^\infty(\Omega).$$

Then there exists a harmonic $u_1 \in C^\infty(\Omega)$ such that $u = u_1$ a.e. in Ω.

Proof. Let $\omega \subseteq \Omega$ be open with compact $\overline{\omega} \subseteq \Omega$, and let $\psi \in C_0^\infty$ with $\psi = 1$ near $\overline{\omega}$. Let

$$w(x,y) = \Delta_y((1-\psi(y))E(x-y)), \qquad x \in \omega, y \in \Omega.$$

Then $w \in C^\infty$, and $y \mapsto w(x,y)$ has compact support: for all $x \in \omega$,

$$w(x,y) = (1-\psi(y))\underbrace{(\Delta E)(x-y)}_{=0} + \underbrace{\cdots}_{\text{has supp} \subseteq \text{supp}(\nabla\psi) \subseteq \Omega}.$$

Let $v(x) = \int u(y)w(x,y)\,dy \in C^\infty(\omega)$. We claim that for all $g \in C_0^\infty(\omega)$, the integral $\int v(x)g(x)\,dx = \int u(x)g(x)\,dx$; this implies that $u = v$ a.e. We have:

$$\begin{aligned}
\int v(x)g(x)\,dx &= \iint u(y)\Delta_y((1-\psi(y))E(x-y))g(x)\,dx\,dy \\
&= \int u(y)\Delta_y\left[(1-\psi(y))\underbrace{\int E(x-y)g(x)\,dx}_{h(y)}\right]dy \\
&= \int u(y)\Delta_y((1-\psi(y))h(y))\,dy
\end{aligned}$$

Here, $h(y) = \int E(x)g(x+y)\,dx \in C^\infty(\mathbb{R}^2)$, where $E \in L^1_{\text{loc}}$, $\psi h \in C_0^\infty(\Omega)$.

$$= \int u(y)\Delta h(y)\,dy - \underbrace{\int u(y)\Delta(\psi h)\,dy}_{=0}$$

E is a fundamental solution to the Lapalacian, so $\Delta h(y) = \int E(x)\Delta g(x+y)\,dx = g(y)$.

$$= \int u(y)y(y)\,dy. \qquad \square$$

Remark 11.1. The argument in the proof only uses that $E \in L^1_{\text{loc}}$ and $E \in C^\infty(\mathbb{R}^2 \setminus \{0\})$. If we replaced the Laplacian by any other operator with a fundamental solution, the same proof would work.

11.2 Perron's method for constructing harmonic functions

Recall Perron's method for $\Omega \subseteq \mathbb{C}$:

Lemma 11.1. *Let $\Omega \subseteq \mathbb{C}$ be open and connected, and let $u : \Omega \to [-\infty, \infty)$ be subharmonic with $u \not\equiv -\infty$. Let $D = \{|x-a| < R\}$ be such that $\overline{D} \subseteq \Omega$, and define*

$$u_D(x) = \begin{cases} u(x) & x \in \Omega \setminus D \\ \frac{1}{2\pi R}\int_{|y|=R} P_R(x-a,y)u(a+y)\,ds(y) & x \in D. \end{cases}$$

Then u_D is subharmonic in Ω, and $u \leq u_D$.

The function u_D is called the **Poisson modification** of u.

Definition 11.1. Let $\Omega \subseteq \mathbb{C}$ be open and connected. A **continuous Perron family** in Ω is a family $\mathcal{F}$ of continuous subharmonic functions $u : \Omega \to [-\infty, \infty)$ such that

1. $u, v \in \mathcal{F} \implies \max(u,v) \in \mathcal{F}$.
2. If $u \in \mathcal{F}$ and D is a disc with $\overline{D} \subseteq \Omega$, then $u_D \in \mathcal{F}$.
3. For each $x \in \Omega$, there is a $u \in \mathcal{F}$ such that $u(x) > -\infty$.

Theorem 11.2 (Perron's method)**.** *Let $\mathcal{F}$ be a continuous Perron family on an open and connected $\Omega \subseteq \mathbb{C}$, and let $u = \sup_{v \in \mathcal{F}} v$ pointwise. Then one of the following statements holds:*

1. *$u(x) \equiv +\infty$ for all $x \in \Omega$.*
2. *u is harmonic in Ω.*

Remark 11.2. The proof is of local nature; it uses only local properties if $v \in \mathcal{F}$, and the maximum principle is only used on small discs in Ω.

Let X be a Riemann surface. We claim that Perron's method works on X.

Definition 11.2. A function $u : X \to [-\infty, \infty)$ is **subharmonic** (resp. **harmonic**) if for every complex chart $\varphi_\alpha : U_\alpha \to V_\alpha$ in some atlas, $u \circ \varphi_\alpha^{-1}$ is subharmonic (resp. **harmonic**) in V_α.

Definition 11.3. A **parametric disc** $D = D_X \subseteq X$ is a set such that there exists a complex chart $\varphi : U \to V$ such that $\overline{D}_X \subseteq U$ and $\varphi(D_X)$ is a Euclidean disc.

Given $u \in SH(X)$, define its **Poisson modification**:

$$u_{D_X}(x) = \begin{cases} u(x) & x \in X \setminus D \\ h(x) & x \in D, \end{cases}$$

where h is a harmonic extension of $u|_{\partial D}$.

The fundamental theorem of Perron's method is valid on X, so we can construct integrable harmonic functions on X.

12 Green's Functions on Riemann Surfaces

12.1 Green's functions on Riemann surfaces

Let X be a Riemann surface. Take $x \in X$, let $z : U \to V$ be a complex chart, and let D be a parametric disc with $x \in D$ and $\overline{D} \subseteq U$ such that $z(x) = 0$. Let $\mathcal{F}$ be a family of continuous subharmonic functions $X \setminus \{x\} \to [-\infty, \infty)$ such that

1. For every $u \in \mathcal{F}$, there is a compact $K \subsetneq X$ such that $u|_{X \setminus K} = 0$.
2. For every $u \in \mathcal{F}$, $u(y) + \log |z(y)|$ is bounded above for y in a neighborhood of X.

$\mathcal{F}$ is a **Perron family** on $X \setminus \{x\}$.

Remark 12.1. The second condition does not depend on the choice of the parametric disc.

Set

$$G_x(y) = \sup_{u \in \mathcal{F}} u(y).$$

Definition 12.1. If $G_x < \infty$, then we say that the harmonic function G_x on $X \setminus \{x\}$ is a **Green's function** for X with pole at $x \in X$.

If $G_x \equiv \infty$, then we say that Green's function does not exist. To give an example where it does exist, first recall the Lindelöf maximal principle:

Theorem 12.1 (Lindelöf maximum principle[4])**.** *Let $\Omega \subseteq \mathbb{C}$ be open and bounded, and let $u \in SH(\Omega)$ be bounded above. If*

$$\limsup_{z \to \zeta} u(z) \leq M \qquad \forall \zeta \in \partial\Omega \setminus F,$$

where F is finite, then $u \leq M$ in all of Ω.

Example 12.1. Let $X = \{|z| < 1\}$. We claim that when $|a| < 1$, Green's function G_a exists, and

$$G_a(z) = \log \left| \frac{1 - \overline{a}z}{z - a} \right|.$$

Let $u \in \mathcal{F}$. Then

$$u(z) - G_a(z) = u(z) + \log \left| \frac{z - a}{1 - \overline{a}z} \right|,$$

which is subharmonic on $D \setminus \{a\}$, bounded above, and equals zero on ∂D. By the Lindelöf maximum principle, $u - G_a \leq 0$ on $D \setminus \{a\}$. We also notice that for every $\varepsilon > 0$, the function $\max(G_a(z) - \varepsilon, 0) \in \mathcal{F}$. The claim follows.

[4]This name is not completely standard but sometimes appears in the literature.

Example 12.2. If $X = \mathbb{C}$, then G_0 does not exist: consider $\max(\log R/|z|, 0)$ for large R.

Proposition 12.1. *Let $x \in X$, and let $z : D \to \mathbb{C}$ be a parametric disc with $z(x) = 0$. Assume that G_x exists. Then $G_x > 0$ on $X \setminus \{x\}$, and $G_x(y) + \log |z(y)|$ extends to a harmonic function on D.*

Proof. Let

$$u_0(y) = \begin{cases} \log \frac{1}{|z(y)|} & y \in D \setminus \{x\} \\ 0 & y \in X \setminus D. \end{cases}$$

Then $u_0 \in \mathcal{F}$. The function u_0 is subharmonic on $X \setminus \{x\}$, as $\max(\log(1/|z|), 0)$ is subharmonic on $\mathbb{C} \setminus \{0\}$. Then $u_0 \geq 0$, so $G_x \geq 0$ on $X \setminus \{x\}$, and $G_x > 0$ on D. By the maximum principle, $G_x > 0$ on $X \setminus \{x\}$.

Let $u \in \mathcal{F}$. Then $u(y) + \log |z(y)|$ is subharmonic in $D \setminus \{x\}$ and bounded above. By the Lindelöff maximum principle,

$$u(y) + \log |z(y)| \leq \sup_{\partial D} u \leq \sup_{\partial D} G_x < \infty, \qquad y \in D \setminus \{x\}.$$

So

$$G_x(y) + \log |z(y)| \leq \sup_{\partial D} G_x, \qquad y \in D \setminus \{x\}.$$

Also,

$$G_x(y) + \log |z(y)| \geq u_0(y) + \log |z(y)| = 0, \qquad y \in D \setminus \{x\}.$$

It follows that the bounded harmonic function $G_x(y) + \log |z(y)|$ extends harmonically to D (the singularity at x is removable). □

Remark 12.2. It follows that $G_x(y) > 0$ is superharmonic on X. This explains why $\mathbb{C}$ does not admit any Green's functions; $-G_x$ would be a bounded subharmonic function on $\mathbb{C}$, but such a function does not exist.

12.2 Uniformization theorem, case 1

Theorem 12.2 (Uniformization, Case 1)**.** *Let X be a simply connected Riemann surface. The following conditions are equivalent:*

1. *$G_x(y)$ exists for some $x \in X$.*
2. *$G_x(y)$ exists for all $x \in X$.*
3. *There exists a holomorphic bijection $\varphi : X \to \{z : |z| < 1\}$.*

Proof. (3) $\implies$ (2): Let $\varphi : X \to \{|z| < 1\}$ be a holomorphic bijection, and let $x \in X$. We can assume that $\varphi(x) = 0$ (by composing φ with a Möbius transformation). Let $v \in \mathcal{F}_x$. Then $v(y) + \log|\varphi(y)|$ is subharmonic on $X \setminus \{x\}$, bounded above, and ≤ 0 far away from x. By the Lindelöf maximum principle, $v(y) + \log|\varphi(y)| \leq 0$ on $X \setminus \{x\}$. So $G_x = \sup_{v \in \mathcal{F}} v$ exists.

(2) $\implies$ (1): This is a special case.

(1) $\implies$ (3): Assume that G_x exists for some $x \in X$. By the proposition, $G_x(y) + \log|z(y)|$ is harmonic in the parametric disc $z : D \to \{|z| < 1\}$ (where $z(x) = 0$). Then there exists $f \in \mathrm{Hol}(D)$ such that $G_x(y) + \log|z(y)| = \mathrm{Re}(f(y))$ for $y \in D$. Let $\varphi(y) := z(y)e^{-f(y)}$. Then $\varphi(x) = 0$, φ is holomorphic, and $|\varphi(y)| = e^{-G_x(y)} < 1$ for all $y \in D$. We claim that φ continues holomorphically to all of X so that this holds globally on X. $\square$

We will prove the last part of this case next time.

13 The Uniformization Theorem

13.1 Uniformization, Case 1

Let's finish the proof of the first case of the Uniformization theorem.

Theorem 13.1 (Uniformization, Case 1)**.** *Let X be a simply connected Riemann surface. The following conditions are equivalent:*

1. *$G_x(y)$ exists for some $x \in X$.*

2. *$G_x(y)$ exists for all $x \in X$.*

3. *There exists a holomorphic bijection $\varphi : X \to \{z : |z| < 1\}$.*

Proof. (1) $\implies$ (3): Let $D \subseteq X$ be a parametric disc with $x \in D$ and $z(x) = 0$. We saw last time that there is a $\varphi_D \in \mathrm{Hol}(D)$ such that $|\varphi_D(y)| = e^{-G_x(y)}$ for all $y \in D$. If $D' \subseteq X$ is a parametric disc such that $x \notin D$, then there exists $\varphi_{D'} \in \mathrm{Hol}(D')$ such that $|\varphi_{D'}(y)| = e^{-G_x(y)}$ for all $y \in D'$: $G|_{D'}$ is harmonic, so $G_x = \mathrm{Re}(f_{D'})$ with $f_{D'}$ holomorphic, and we can take $\varphi_{D'}(y) = e^{-f_{D'}(y)}$. On $D \cap D'$, $\varphi_D/\varphi_{D'}$ is holomorphic with modulus 1. So $\varphi_D/\varphi_{D'} = e^{i\theta}$ for some θ.

Let γ be a path in X with $\gamma(0) = x$. Then, by compactness, there is a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ and parametric discs D_j, $1 \leq j \leq n$, such that $\gamma([t_{j-1}, t_j]) \subseteq D_j$. It follows that φ_D can be continued analytically along all paths in X starting at x. By the monodromy theorem, there is a globally defined holomorphic function $\varphi \in \mathrm{Hol}(X)$ such that $|\varphi(y)| = e^{-G_x(y)}$ for all $y \in X$.

We claim that φ is injective. We have that $\varphi(x) = 0$, and if $\varphi(y) = \varphi(y) = \varphi(x) = 0$, then $y = x$ (since G_x is only infinite at x). Let $z \in X$ with $z \neq x$. Then $|\varphi(z)| < 1$. Consider

$$\varphi_1(y) = \frac{\varphi(y) - \varphi(z)}{1 - \overline{\varphi(z)}\varphi(y)}.$$

Then $\varphi_1 \in \mathrm{Hol}(X)$, and $|\varphi_1| < 1$. Take $v \in \mathcal{F}_z$, the Perron family used to construct G_z. The function $v(y) + \log|\varphi_1(y)|$ is subharmonic on $X \setminus \{z\}$, bounded above, and ≤ 0 far away. By the Lindelöf maximum principle, $v(y) + \log|\varphi_1(y)| \leq 0$ on $X \setminus \{z\}$. So G_z exists, and $G_z(y) + \log|\varphi_1(y)| < 0$. For $y = x$, we get

$$G_z(x) \leq -\log|\varphi_1(x)| = \log|\varphi(z)| = G_x(z).$$

Switching the roles of x and z, we get[5]

$$G_z(x) = G_x(z).$$

[5] This symmetry of the Green's function is actually true in general, but we will not visit that fact now.

The function $G_z(y) + \log|\varphi_1(y)| \leq 0$ is subharmonic for $y \neq z$, and when $y = x$, we have

$$G_z(x) + \log|\varphi_1(x)| = G_x(z) + \log|\varphi(z)| = 0.$$

By the maximum principle, we get

$$G_z(y) = -\log|varphi_1(y)|, \qquad y \neq z.$$

If $\varphi(w) = \varphi(z)$, then $\varphi_1(w) = 0$. So $G_z(w) = \infty$, which means $w = z$.

We have that $\varphi : X \to D = \{|z| < 1\}$ is holomorphic and injective. We do not actually need to prove surjectivity because of the following trick.[6] $\varphi(X) \subseteq D$ is open and simply connected. By the Riemann mapping theorem, there is a holomorphic bijection $\psi : \varphi(X) \to D$. So the map $\psi \circ \varphi \in \mathrm{Hol}(X)$ works. □

Remark 13.1. This is sometimes called the hyperbolic case since D admits a hyperbolic metric. So we have shown that every simply connected manifold that carries a Green's function is conformally equivalent to a space with a hyperbolic metric.

13.2 Uniformization, Case 2

Theorem 13.2 (Uniformization, Case 2)**.** *Let X be a simply connected Riemann surface for which Green's function does not exist. If X is compact, then there is a holomorphic bijection $X \to \hat{\mathbb{C}}$. If X is not compact, there is a holomorphic bijection $X \to \mathbb{C}$.*

The main idea in the proof is to show the existence of a **dipole Green's function**.

Example 13.1. Consider $\log 1/|z|$ on the Riemann sphere. This has singularities of opposite signs at 0 and ∞.

Lemma 13.1 (existence of a dipole Green's function)**.** *Let X be a Riemann surface, let $x_1, x_2 \in X$ be distinct, and let $z_j : D_j \to \{|z| < 1\}$ for $j = 1, 2$ be parametric discs such that $z_j(x_j) = 0$, snd $\overline{D}_1 \cap \overline{D}_2 = \varnothing$. Then there is a functio n$G_{x_1,x_2}(y)$ which is harmonic on $X \setminus \{x_1, x_2\}$ such that $G_{x_1,x_2}(y) + \log|z_1(y)|$ is harmonic in D_1 and $G_{x_1,x_2}(y) - \log|z_2(y)|$ is harmonic in D_2. Furthermore,*

$$\sup_{y \in X \setminus (D_1 \cup D_2)} G_{x_1,x_2}(y) < \infty.$$

Assuming this lemma, which we will prove later, we can finish the proof of the Uniformization theorem.

[6] The map is actually surjective, but it would take some more work to prove.

Proof. Let G_{x_1,x_2} a dipole Green's function for $x_1 \neq x_2 \in X$. Arguing as in the proof of Case 1, we see that there is a $\varphi \in \mathrm{Hol}(X, \hat{\mathbb{C}})$ (i.e. a meromorphic function on X) such that

$$|\varphi(y)| = e^{-G_{x_1,x_2}(y)}, \qquad y \in X.$$

Then φ has a unique zero at x_1 at x_1 and a unique simple pole at z_2.

Assume that $\varphi : X \to \mathbb{C}$ is injective. Then consider $\varphi(X) \subseteq \hat{\mathbb{C}}$, which is simply connected. If $\hat{\mathbb{C}} \setminus \varphi(X)$ contains more than a single point, composing with a Möbius transformation which sends the point to ∞, we get an injective, holomorphic map from X to a subset of $\mathbb{C}$. By the Riemann mapping theorem, we get a holomorphic bijection to D; however, we assumed no Green's function exists, so we have a contradiction. So we must either have $\varphi(X) = \mathbb{C}$ (after composing with a Möbius transformation) or $\varphi(X) = \hat{\mathbb{C}}$. $\square$

Next time, we will show that φ is injective, to complete the proof.

14 Uniformization Case 2 and Green's Functions Away From a Disc

14.1 Uniformization, Case 2 (cont.)

Last time, we were finishing our proof of the Uniformization theorem.

Theorem 14.1 (Uniformization, Case 2)**.** *Let X be a simply connected Riemann surface for which Green's function does not exist. If X is compact, then there is a holomorphic bijection $X \to \hat{\mathbb{C}}$. If X is not compact, there is a holomorphic bijection $X \to \mathbb{C}$.*

Proof. If G_{x_1,x_2} is a dipole Green's function, then there is a $\varphi \in \mathrm{Hol}(X, \hat{\mathbb{C}})$ such that $|\varphi(y)| = e^{-G_{x_1,x_2}(y)}$, $\varphi(x_1) = 0$, and $\varphi(x_2) = \infty$ (a simple pole). We only need to show that φ is injective on X. Let $x_0 \in X \setminus \{x_1, x_2\}$. The dipole Green's function $G_{x_0,x_2}(y)$ exists, then there is a $\varphi_0 \in \mathrm{Hol}(X, \hat{\mathbb{C}})$ such that $|\varphi_0(y)| = e^{-G_{x_0,x_2}(y)}$ for $y \in X$. Consider the function

$$f(y) = \frac{\varphi(y) - \varphi(x_0)}{\varphi_0(y)},$$

which is holomorphic away from x_0, x_2. The singularities at x_0, x_2 are removable, so $f \in \mathrm{Hol}(X)$.

Now

$$\sup_{y \in X \setminus (D_1 \cup D_2)} < \infty \implies |f(y)| \leq e^{G_{x_0,x_2}(y)}(e^{-G_{x_1,x_2}(y)} + C),$$

so f is bounded away from x_0, x_1, x_2. Since f is holomorphic at these 3 points, f is bounded on all of X. Say $|f(y)| \leq M$. Let $v \in \mathcal{F}_{x_1}$ be a Perron amily for G_{x_1}. Then

$$v(y) + \log\left|\frac{f(y) - f(x_1)}{2M}\right|, \qquad y \in X \setminus \{x_1\}$$

by the Lindelöf maximum principle. Since $\sup_{v \in \mathcal{F}_{x_1}} v(y) = \infty$ for all y, we get $f(y) = f(x_1)$ for all $y \in X$.

We get that

$$\frac{\varphi(y) - \varphi(x_0)}{\varphi_0(y)} = \frac{\varphi(x_1) - \varphi(x_0)}{\varphi_0(x_1)} = -\frac{\varphi(x_0)}{\varphi_0(x_1)} \notin \{0, \infty\}.$$

In particular, $\varphi \neq \varphi(x_0)$ unless $\varphi_0(y) = 0$. This is when $y = x_0$. Thus, φ is injective on $X \setminus \{x_1, x_2\}$ and hence on X. $\square$

14.2 Existence of a Green's function away from a disc

It now remains to prove the existence of a dipole Green's function. We need the following fact.

Theorem 14.2. *Let X_0 be a Riemann surface, and let $D_0 \subseteq X_0$ be a parametric disc. Set $X = X_0 \setminus \overline{D}_0$. Then for all $x \in X$, a Green's function $G_x(y)$ on X exists.*

Given this construction, we can produce a dipole Green's function by taking the difference of Green's functions G_{x_1} and G_{x_2} for $x_1, x_2 \notin \overline{D}_0$. Then we can shrink the size of the disc to try to get a dipole Green's function on all of X_0.

Proof. Let $x \in X$, and let $S \subseteq X$ be a parametric disc $D \subseteq X$ with $x \in D \cong \{|z| < 1\}$ and $z(x) = 0$. When $0 < r < 1$, let $rD = \{y \in D : |z(y)| < r\}$. Let $v \in \mathcal{F}_x$, a Perron family on X. Then

$$v(y) + \log|z(y)| \leq \sup_{\partial D} V, \qquad y \in D, y \neq x$$

by the Lindelöf maximum principle. In particular,

$$\sup_{y \in \partial(rD)} v(y) + \log(r) \leq \sup_{\partial D} v.$$

Idea: We want to solve the Dirichlet problem[7] on $X \setminus \overline{rD} = X_0 \setminus (\overline{D}_0 \cup \overline{rD})$:

$$\Delta u = 0 \text{ on } X \setminus \overline{rD}, \qquad u|_{\partial(rD)} = 1, \qquad u|_{\partial D_0} = 0.$$

We will use Perron's method. Let $\mathcal{F}$ be the collection of us which are subharmonic on $X \setminus \overline{rD}$, $u = 0$ far away, and such that

$$\limsup_{y \to \zeta} u(y) \leq 1 \qquad \forall \zeta \in \partial(rD),$$

$$\limsup_{y \to \alpha} u(y) \leq 0 \qquad \forall \alpha \in \partial D_0.$$

For all $u \in \mathcal{F}$, $u \leq 1$, so by the Perron theorem,

$$\omega(y) = \sup_{v \in \mathcal{F}} v(y)$$

is harmonic on $X \setminus \overline{rD}$.

Any point $\xi \in \partial D_0 \cup \partial(rD)$ is a **regular point** for the Dirichlet problem in the sense that there is a local barrier at ξ: Recall that h is a **local barrier** at $\xi \in \partial\Omega$ (where $\Omega \subseteq \mathbb{C}$ is open and connected) if

1. h is defined and subharmonic on $\Omega \cap V$ for some neighborhood B of ξ.
2. $h(z) < 0$ in $\Omega \cap V$
3. For $z \in \Omega$ $h(z) \to 0$ as $z \to \xi$.

[7] We have not formally defined the Laplacian on a Riemann surface, but this should at least motivate the rest of the proof.

If $\partial\Omega \in C^1$, then any $\xi \in \partial\Omega$ is a regular point. By Perron's theorem, it follows that $\omega = \sup v$ extends continuously to $\partial(rD) \cup \partial D_0$. So we have a harmonic ω on $X \setminus \overline{rD}$ such that $\omega|_{\partial(rD)} = 1$ and $\omega|_{\partial D_0} = 0$. We have that $0 \leq \omega \leq 1$, and by the maximum principle, $0 < \omega < 1$ on $X \setminus \overline{rD}$.

Let us go back to $v \in \mathcal{F}_x$:

$$\sup_{y \in \partial(rD)} v(y) + \log(r) \leq \sup_{\partial D} v.$$

Consider the subharmonic function on $X \setminus \overline{rD}$

$$v - \left(\sup_{\partial(rD)} v\right) \omega.$$

By the maximum principle, this function is ≤ 0. So

$$v \leq \left(\sup_{\partial D} v\right) w,$$

which gives us that

$$\sup_{\partial D} v \leq \left(\sup_{\partial(rD)} v\right) \underbrace{\sup_{\partial D} \omega}_{=1-\delta}.$$

Combining this with our previous bound on v gives

$$\delta \sup_{\partial(rD)} \leq \sup_{\partial(rD)} v - \sup_{\partial D} v,$$

so

$$\delta \sup_{\partial(rD)} + \log(r) \leq 0.$$

We get that

$$\sup_{\partial(rD)} \leq \frac{1}{\delta} \log\left(\frac{1}{r}\right), \qquad \forall v \in \mathcal{F}_x$$

Thus, $\sup_{v \in \mathcal{F}} v \neq \infty$, and G_x exists. $\square$

Remark 14.1. The function ω is called the **harmonic measure** of $\partial(rD)$ in the region $X \setminus \overline{rD}$.

15 Existence of a Dipole Green's Function

15.1 Symmetry of Green's functions

Proposition 15.1 (symmetry of Green's functions)**.** *Let X be a Riemann surface such that G_x exists for some $x \in X$. Then G_y exists for any y, and $G_x(y) = G_y(x)$.*

We have already proven this when X is simply connected.

Proof. Idea: Let $\tilde{X}$ be a universal covering space of X. On $\tilde{X}$, $G_{\tilde{z}}$ exists for all $\tilde{z} \in p^{-1}(x)$, where $p : \tilde{X} \to X$ is a covering map. So $\tilde{X} = D$, and

$$G_{\tilde{z}}(\tilde{y}) = \log \left| \frac{1 - \overline{\tilde{z}}\tilde{y}}{\tilde{y} - \tilde{z}} \right|$$

is symmetric. $\square$

Remark 15.1. It follows that any Riemann surface is second countable (Rado's theorem). Take X, and remove a parametric disc. Then the rest of the space has a Green's function, so it is covered by a disc, which is second countable.

15.2 Existence of a dipole Green's function

Theorem 15.1 (existence of a dipole Green's function)**.** *Let X be a Riemann surface, and let $x_1 \neq x_2 \in X$. Let $z_j : D_j \to \{|z| < 1\}$ be parametric discs such that $z_j(x_j) = 0$ and $\overline{D}_1 \cap \overline{D}_2 = \varnothing$. Then there exists a harmonic G_{x_1,x_2} on $X \setminus \{x_1, x_2\}$ such that $G_{x_1,x_2} + \log|z_1(y)|$ is harmonic in D_1, $G_{x_1,x_2} + \log|z_2(y)|$ is harmonic in D_2, and $\sup_{X \setminus (D_1 \cup D_2)} |G_{x_1,x_2}| < \infty$.*

Proof. Let $D_0 \subseteq X$ be a parametric disc $z_0 : D_0 : \{|z| < 1\}$ with $z_0(x_0) = 0$ and $\overline{D}_0 \cap \overline{D}_j = \varnothing$ for $j = 1, 2$. For $0 < t < 1$, let $tD_0 = \{y \in D_0 : |z_0(y)| < t\}$. Let $X_t = X \setminus \overline{tD_0}$. We know that Green's function $G_{X_t}(x_1, y)$ exists for all $y \in X_t \setminus \{x_1\}$ and for all t. Let $0 < r < 1$. Let $v \in \mathcal{F}_{x_1}$, the Perron family on X_t used to construct $G_{X_t}(x_1, y)$. When $y \in X_t \setminus \overline{rD_1}$,

$$v(y) \leq \sup_{\partial(rD_1)} v$$

by the maximum principle. Taking the sup over all $v \in \mathcal{F}_{x_1}$,

$$G_{X_y}(x_1, y) \leq \sup_{\partial(rD_1)} G_{X_t}(x_1, y) =: M(t).$$

On the other hand, we have shown last time that

$$\sup_{\partial(rD_1)} v + \log(r) \leq \sup_{\partial D_1} v$$

(by applying the maximum principle to $v(y) + \log|z_1(y)|$ in D_1). We get

$$M(t) + \log(r) \leq \sup_{\partial D_1} G_{X_t}(x_1, y).$$

Consider the function

$$u_t(y) = M(t) - G_{X_t}(x_1, y), \qquad y \in X_t \setminus \overline{rD_1}$$

Then $u_t(y) \geq 0$ and is harmonic. There exists a $y_0 \in \partial D_1$ such that $u_t(y_0) \leq \log(1/r)$. We want to apply Harnack's principle to u_t: Let $K \subseteq X_1 \setminus \overline{rD_1}$ be compact such that $\overline{D}_2 \subseteq K_1$ and $\partial D_1 \subseteq K$. By Harnack's inequality,

$$\frac{\sup_K u_t}{\inf_K u_t} \leq C(K, r),$$

where $C(K, r)$ is a geometric constant independent of t. So

$$u_t(y) \leq C, \qquad y \in K,$$

uniformly in t. So

$$|G_{X_t}(x_1, y) - G_{X_t}(x_1, x_2)| = |u_t(y) - u_t(x_2)| \leq 2C.$$

Similarly,

$$|G_{X_t}(x_2, y) - G_{X_t}(x_2, x_1)| \leq 2C, \qquad y \in K', K' \supseteq \overline{D}_1 \cup \partial D_2.$$

By the symmetry of Green's functions, $G_{X_t}(x_2, x_1) = G_{X_t}(x_1, x_2)$. So we get

$$|G_{X_t}(x_1, y) - G_{X_t}(x_2, y)| \leq C$$

uniformly in t for $y \in \partial D_1 \cup \partial D_2$.

We also want uniform control on G_t on $X_t \setminus (D_1 \cup D_2)$: Let $v \in \mathcal{F}_{x_1}$. Then $v(y) - G_{X_t}(x_2, y)$ is subharmonic for $y \in X_t \setminus \overline{D}_1$, so

$$v(y) - G_{X_t}(x_2, y) \leq \sup_{\partial D_1}(v - G_{X_t}(x_2, y)) \leq C$$

by the maximum principle. So

$$\underbrace{G_{X_t}(x_1, y) - G_{X_t}(x_2, y)}_{:= G_t(y, x_1, x_2)} \leq C$$

on $X_t \setminus D_1$. Similarly,

$$\inf_{y \in X_t \setminus D_2} G_t = -\sup_{X_t \setminus D_2} -G_t \geq C,$$

so we get

$$\sup_{X_t \setminus (D_1 \cup D_2)} |G_t| \leq C,$$

uniformly in t. In D_j, $j = 1, 2$, $G_t(y, x_1, x_2) + \log|z_1(y)|$ is harmonic in D_1. By the maximum principle applied in D_1,

$$|G_t(y, x_1.x_2) + \log|z_1(y)|| \leq C, \qquad y \in D_1,$$

uniformly in t. Similarly,

$$|G_t(y, x_1, x_2) - \log|z_2(y)|| \leq C, \qquad y \in D_2,$$

uniformly in t.

These three uniform inequalities give us the following: Let $K \subseteq X \setminus \{x_1, x_2, x_0\}$ be compact. By normal families and Rado's theorem, there exists a sequence $t_n \to 0$ and G harmonic on $X \setminus \{x_0, x_1, x_2\}$ such that $G_{t_n} \to G$ locally uniformly on $X \setminus \{x_0, x_1, x_2\}$. The first inequality gives us that G is bounded in $D_0 \setminus \{x_0\}$; so G extends harmonically to D_0. Similarly,

$$|G(y) + \log|z_1(y)| \leq C \text{ in } D_1 \implies G + \log|z_1| \text{ is harmonic in } D_1,$$

$$|G(y) + \log|z_2(y)| \leq C \text{ in } D_2 \implies G + \log|z_2| \text{ is harmonic in } D_2.$$

So G is a dipole Green's function. □

16 Consequences of the Uniformization Theorem

16.1 Deck transformations

We have shown the Uniformization theorem.

Theorem 16.1 (Uniformization)**.** *Let X be a simply connected Riemann surface.*

1. *If Green's function exists for X, then there is a holomorphic bijection $X \to D$.*
2. *If X is compact, then $X \cong \hat{\mathbb{C}}$.*
3. *If X is not compact and if Green's function does not exist, then $X \cong \mathbb{C}$.*

What does this say about non-simply connected Riemann surfaces?

Let X be a connected topological manifold. Let $\tilde{X}$ be the universal covering space of X with covering map $p : \tilde{X} \to X$.

Definition 16.1. We say that a homeomorphism $\varphi : \tilde{X} \to \tilde{X}$ is a **deck transformation** if $p \circ \varphi = p$.

Proposition 16.1. *The set of deck transformations is a group G which acts transitively on the fibers: if $\tilde{x}, \tilde{y} \in \tilde{X}$ such that $p(\tilde{x}) = p(\tilde{y})$, there is a unique $\varphi \in G$ such that $\varphi(\tilde{x}) = \tilde{y}$.*

Proof. The lifting criterion applied to p gives a unique $\varphi : \tilde{X} \to \tilde{X}$ such that $p \circ \varphi = p$ and $\varphi(\tilde{x}) = \tilde{y}$.

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ \tilde{X} \arrow[ur, dashed, "\varphi"] \arrow[r, "p"'] & X \end{tikzcd}$$

φ is a homeomorphism because there is a continuous map $\psi : \tilde{X} \to \tilde{X}$ such that $P \circ \tilde{\psi} = p$ and $\psi(\tilde{y}) = \tilde{x}$. So $p \circ \varphi \circ \psi = p$ and $\varphi(\psi(\tilde{y})) = \tilde{y}$. So $\varphi \circ \psi = 1$ by the uniqueness of lifts. So φ is a deck transformation. $\square$

Proposition 16.2. *The group G acts on $\tilde{X}$ freely: for all $\varphi \in G$ with $\varphi \neq 1$, φ has no fixed points. Also, the orbits $G\tilde{x} = \{\varphi(\tilde{x}) : \varphi \in G\} = p^{-1}(p(\tilde{x}))$ are discrete, as p is a cover.*

Corollary 16.1. *The space of orbits $\tilde{X}/G$ is naturally identified with X, also topologically if $\tilde{X}/G$ is equipped with the quotient topology: $O \subseteq \tilde{X}/G$ is open iff $\pi^{-1}(P) \subseteq \tilde{X}$ is open, where $\pi : \tilde{X} \to \tilde{X}/G$ is the quotient map $\tilde{x} \mapsto G\tilde{x}$.*

16.2 Partial classification of Riemann surfaces

Let X be a Riemann surface. Then $\tilde{X}$ is a Riemann surface, and $p : \tilde{X} \to X$ is holomorphic. So every $\varphi \in G$ is holomorphic: $G \subseteq \mathrm{Aut}(\tilde{X}) = \{\text{holomorphic bijections } \tilde{X} \to \tilde{X}\}$. We have $X = \tilde{X}/G$, where by uniformization, $\tilde{X} = \hat{\mathbb{C}}$, $\mathbb{C}$, or D.

1. $\tilde{X} = \hat{\mathbb{C}}$: We have that $G \subseteq \mathrm{Aut}(\hat{\mathbb{C}}) = \{\sigma : \sigma(z) = \frac{az+b}{cz+d}, ad - bc \neq 0\}$. Every $\sigma \in \mathrm{Aut}(\mathbb{C})$ has a fixed point, so $G = \{1\}$. We get that if X is a Riemann surface with $\hat{\mathbb{C}}$ has the universal covering space, $X = \mathbb{C}$.

2. $\tilde{X} = \mathbb{C}$: We have that $G \subseteq \mathrm{Aut}(\mathbb{C}) = \{\sigma : \sigma(z) = az + b, a \neq 0, b \in \mathbb{C}\}$. The elements of G have no fixed points, so $a = 1$. We get that $G \subseteq \{\sigma : \sigma(z) = z + b, b \in \mathbb{C}\}$, the complex translations. G acts with discrete orbits, so (by a fact we will not prove here[8]) one of the following holds:

   (a) $G = \{1\}$, so $X \cong \mathbb{C}$.

   (b) $G = \{\sigma : \sigma(z) = z + n\gamma, n \in \mathbb{Z}\}$ for some $\gamma \in \mathbb{C} \setminus \{0\}$. We have a natural isomorphism $X \cong \mathbb{C}/\{z \mapsto z + n\gamma\} \cong \mathbb{C} \setminus \{0\}$ via $[z] \mapsto e^{2\pi i z/\gamma}$.

   (c) $G = \{\sigma : \sigma(z) = n\gamma + m\delta + z, n, m \in \mathbb{Z}\}$, where $\gamma, \delta \in \mathbb{C}$ are linearly independent over $\mathbb{R}$. In this case, X is isomorphic to the complex torus.

   Thus, if X is a Riemann surface with $\tilde{X} = \mathbb{C}$, then $X \cong \mathbb{C}$, $\mathbb{C} \setminus \{0\}$, or a complex torus.

3. $\tilde{X} = D$. Then $X \cong D/G$, where $G \subseteq \mathrm{Aut}(D)$ acts freely. Such subgroups are called **Fuchsian groups**. This is the general case.

16.3 Examples of applications

Example 16.1. Let M be a compact Riemann surface, and assume that there is some $f \in \mathrm{Hol}(\mathbb{C}, M)$ which is non-constant. What can be said about M? Lift f to the universal covering space:

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ \mathbb{C} \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

Then $\tilde{f}$ is non-constant, so $\tilde{M} \neq D$. If $\tilde{M} = \hat{\mathbb{C}}$, then either $M \cong \hat{\mathbb{C}}$ or $\tilde{M} = \mathbb{C}$ and $M \cong$ a torus.

[8] This fact has nothing to do with Riemann surfaces. We have a discrete group acting on a real vector space, so the number of generators should be $\leq$ the dimension of the vector space.

Theorem 16.2 (Picard's little theorem)**.** *Let* $f \in \mathrm{Hol}(\mathbb{C})$ *be such that* $0, 1 \notin f(\mathbb{C})$*. Then* f *is constant.*

Proof. We can lift f:

$$\begin{tikzcd} & D \arrow[d, "p"] \\ \mathbb{C} \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & \mathbb{C} \setminus \{0, 1\} \end{tikzcd}$$

By Liouville's theorem, $\tilde{f}$ is constant. So f is constant. □

This is the end of our discussion of Riemann surfaces. If you are interested in learning more, here are books which have a modern approach to analysis on Riemann surfaces:

- S. Donaldson, Riemann surfaces.
- D. Varolin, Riemann surfaces by way of complex analytic geometry.

17 Introduction to Several Complex Variables

17.1 Holomorphic functions of several complex variables

Definition 17.1. Let $\Omega \subseteq \mathbb{C}^n$ be open, and let $f : \Omega \to \mathbb{C}$ be a function $f = f(z_1, \dots, z_n) = f(x_1, y_1, \dots, x_n, y_n)$, where $z_j = x_j + y_j$. We say that f is **holomorphic** in Ω if $f \in C^1(\Omega)$ and if for every j, $z_j \mapsto f(z_1, \dots, z_j, \dots, z_n)$ where it is defined.

Define

$$\frac{\partial f}{\partial \overline{z}_j} = \frac{1}{2}\left(\frac{\partial f}{\partial x_j} + i\frac{\partial f}{\partial y_j}\right)$$

for $1 \leq j \leq n$. Then f is holomorphic if and only if $f \in C^1(\Omega)$ and $\frac{\partial f}{\partial \overline{z}_j} = 0$ for all j.

Define also

$$\frac{\partial f}{\partial z_j} = \frac{1}{2}\left(\frac{\partial f}{\partial x_j} + \frac{1}{i}\frac{\partial f}{\partial y_j}\right).$$

For all $f \in C^1(\Omega)$,

$$df = \underbrace{\sum_{j=1}^{n} \frac{\partial f}{\partial z_j}\, dz_j}_{=:\partial f} + \underbrace{\sum_{j=1}^{n} \frac{\partial f}{\partial \overline{z}_j}\, d\overline{z}_j}_{=:\overline{\partial} f}.$$

So f is holomorphic iff $ba\partial f = 0$.

Example 17.1. Let $f \in L^1(\mathbb{R}^n)$ be such that $f = 0$ for large $|x|$. Then the Fourier transform

$$\widehat{f}(\xi) = \int f(x)e^{-ix\cdot\xi}\, dx, \qquad \xi \in \mathbb{R}^n$$

extends to the entire function

$$\widehat{f}(\zeta) = \int f(x)e^{-ix\cdot\zeta}\, dx, \qquad \zeta \in \mathbb{C}^n,$$

where $x \cdot \zeta = \sum_j x_j\zeta_j$ (in particular, there are no complex conjugates involved).

Remark 17.1. The space of holomorphic functions, $\mathrm{Hol}(\Omega)$ is a ring.

17.2 Cauchy's integral formula in a polydisc

What is the analogue of a disc in $\mathbb{C}^n$? We could try Euclidean balls, but this turns out to be more complicated.

Definition 17.2. A **polydisc** $D \subseteq \mathbb{C}^n$ si a set of the form $D = D_1 \times \cdots \times D_n$, where each D_j is an open disc in $\mathbb{C}$. The **boundary** is $\partial D = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n : \exists j \text{ s.t. } z_j \in \partial D_j\}$. The **distinguished boundary** of D is $\partial_0 D = \{z \in \mathbb{C}^n : z_j \in \partial D_j\ \forall j\}$.

Theorem 17.1 (Cauchy's integral formula in a polydisc)**.** *Let $D = D_1 \times \cdots D_n$ be a polydisc, let $f \in C(\overline{D})$ be such that f is separately holomorphic*[9] *in $z_j \in D_j$ for all j. Then*

$$f(z) = \frac{1}{(2\pi i)^n} \int_{\partial_0 D} \frac{f(\zeta)}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)} \, d\zeta_1 \cdots d\zeta_n.$$

(The integral can be defined by parametrizing $\partial_0 D$: for $D_j = \{|z_j - \alpha_j| < r_j\}$, let $\zeta_j(t) = \alpha_j + r_j e^{it_j}$, $0 \leq t_j \leq 2\pi$.)

Proof. Proceed by induction on n. When $n = 1$, this is the usual Cauhy's integral formula. Suppose the formula holds for $n - 1$. Write $D = D(\alpha_1, r_1) \times D'$, where $D(\alpha_1, r_1) \subseteq \mathbb{C}$ and $D' \subseteq \mathbb{C}^{n-1}$. For every $z \in D(\alpha_1, r_1)$,

$$f(z, z') = \frac{1}{(2\pi i)^{n-1}} \int_{\partial_0 D'} \frac{f(z, \zeta')}{(\zeta_2 - z_2) \cdots (\zeta_n - z_n)} \, d\zeta'.$$

By Cauchy's integral formula and the fact that $f \in C(\overline{D})$,

$$\begin{aligned} f(z, \zeta') &= \frac{1}{2\pi i} \int_{\partial D(\alpha_1, r_1)} \frac{f(\zeta, \zeta')}{\zeta - z} \, d\zeta \\ &= \frac{1}{2\pi i} \int_{\partial D(\alpha_1, r_1)} \frac{1}{\zeta - z} \left[\frac{1}{(2\pi i)^{n-1}} \int_{\partial_0 D'} \frac{f(z, \zeta')}{(\zeta_2 - z_2) \cdots (\zeta_n - z_n)} \, d\zeta' \right] d\zeta \\ &= \frac{1}{(2\pi i)^n} \int_{\partial_0 D} \frac{f(\zeta)}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)} \, d\zeta_1 \cdots d\zeta_n. \end{aligned}$$

The result follows. □

Corollary 17.1. *Let f satisfy the assumptions in the theorem. Then $f \in C^\infty(D)$, and therefore, $f \in \mathrm{Hol}(D)$.*

Corollary 17.2. *Let $\Omega \subseteq \mathbb{C}$ be open, and let $f \in C(\Omega)$ be separately holomorphic. Then $f \in \mathrm{Hol}(\Omega)$.*

Proof. Take a polydisc D with $\overline{D} \subseteq \Omega$ around each point. □

17.3 Local uniform convergence of holomorphic functions

Theorem 17.2. *Let $u_k \in \mathrm{Hol}(\Omega)$ be such that $u_k \to u$ locally uniformly in Ω. Then $u \in \mathrm{Hol}(\Omega)$, and for every α, $\partial^\alpha u_k \to \partial^\alpha u$ locally uniformly. Here, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ is a multiindex, and $\partial^\alpha = \partial_{z_1}^{\alpha_1} \cdots \partial_{z_n}^{\alpha_n}$.*

Proof. Let D be a polydisc with $\overline{D} \subseteq \Omega$. Then

$$u_k(z) = \frac{1}{(2\pi i)^n} \int_{\partial_0 D} \frac{u_k(\zeta)}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)} \, d\zeta, \qquad z \in D.$$

It follows that $u \in \mathrm{Hol}(\Omega)$, and $\partial^\alpha u_k \to \partial^\alpha u$ uniformly in a neighborhood of the center of D for all α. □

[9] In particular, we are not assuming that f is holomorphic because we do not assume that $f \in C^1$.

17.4 Cauchy's estimates

Let $D \subseteq \mathbb{C}^n$ be a polydisc, let $u \in C(\overline{D}) \cap \mathrm{Hol}(D)$, and write

$$u(z) = \frac{1}{(2\pi i)^n} \int_{\partial_0 D} \frac{u(\zeta)}{(\zeta - z)^E} \, d\zeta.$$

Here, when α is a multiindex, write $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$, and denote $E(1, \ldots, 1)$. Also, when α is a multiindex, denote $\alpha! := \alpha_1! \cdots \alpha_n!$. Then for all α,

$$\partial^\alpha u(z) = \frac{\alpha!}{(2\pi i)^n} \int_{\partial_0 D} \frac{u(\zeta)}{(\zeta - z)^{E+\alpha}} \, d\zeta.$$

We then have Cauchy's estimates:

Theorem 17.3 (Cauchy's estimates)**.** *Let $D \subseteq \mathbb{C}^n$ be a polydisc centered at 0, and let $u \in C(\overline{D}) \cap \mathrm{Hol}(D)$. Then*

$$|\partial^\alpha u(0)| \leq \alpha! \frac{M}{r^\alpha}, \qquad M = \sup_{\partial_0 D} |u|.$$

Proof. By taking derivatives in the Cauchy integral formula as above, we get

$$|\partial^\alpha u(0)| \leq \frac{\alpha!}{(2\pi i)^n} \frac{M (2\pi i)^n r^E}{r^{E\alpha}} = \alpha! \frac{M}{r^\alpha}.$$

□

17.5 Analyticity of holomorphic functions

Theorem 17.4. *Let $D \subseteq \mathbb{C}^n$ be a polydisc centered at 0, and let $f \in \mathrm{Hol}(D)$. We have, with normal convergence in D:*

$$f(z) = \sum_\alpha \frac{\partial^\alpha f(0)}{\alpha!} z^\alpha.$$

Here, normal convergence means that $\sum u_j$ converges normally in Ω ($\sum \sup_K |u_j| < \infty$) for all compact $K \subseteq \Omega$.

18 Analyticity, Maximum Principle, and Hartogs' Lemma

18.1 Analyticity of holomorphic functions

Last time, we defined holomorphic functions of several complex variables: if $\Omega \subseteq \mathbb{C}^n$ is open, then $f \in \mathrm{Hol}(\Omega)$ if $f \in C^1(\Omega)$ and $\frac{\partial f}{\partial \bar{z}_j} = 0$ for all j.

Theorem 18.1. *Let $D \subseteq \mathbb{C}^n$ be a polydisc centered at 0, and let $f \in \mathrm{Hol}(D)$. We have, with normal convergence in D:*

$$f(z) = \sum_{\alpha} \frac{\partial^\alpha f(0)}{\alpha!} z^\alpha.$$

Here, normal convergence means that $\sum u_j$ converges normally in Ω ($\sum \sup_K |u_j| < \infty$) for all compact $K \subseteq \Omega$.

Proof. Let $D' = \{|z_j| < r'_j\}$ for $1 \leq j \leq n$, where $0 < r'_j < r_j$ (and $D = D_1 \times \cdots \times D_n$, $D_j = \{|z_j| < r_j\}$). Then, by Cauchy's integral formula,

$$f(z) = \frac{1}{(2\pi i)^n} \int_{\partial_0 D'} \frac{f(\zeta)}{(\zeta - z)^E\, d\zeta, \qquad E = (1, \dots, 1}.$$

If $|\zeta_j| = r'_j$ and $|z_j| \leq r''_j < r'_j$, then

$$\frac{1}{\zeta_j - z_j} = \frac{1}{\zeta_j} \sum_{k=0}^{\infty} \left(\frac{z_j}{\zeta_j} \right)^k.$$

Then

$$\frac{1}{(\zeta - z)^E} = \sum_{\alpha \in \mathbb{N}^n} \frac{z^\alpha}{\zeta^{\alpha + E}}, \qquad (\zeta, z) \in \partial_0 D' \times D'$$

with normal convergence. We get

$$f(z) = \sum_{\alpha} z^\alpha \frac{1}{(2\pi i)^n} \int_{\partial_0 D'} \frac{f(\zeta)}{\zeta^{\alpha+E}}\, d\zeta = \sum_{\alpha} z^\alpha \frac{\partial^\alpha f(0)}{\alpha!}.$$

As $\overline{D'} \subseteq D$ is arbitrary, the result follows. $\square$

Corollary 18.1. *Let $\Omega \subseteq \mathbb{C}^n$ be open and connected. If $f \in \mathrm{Hol}(\Omega)$ and $\partial^\alpha f(z_0) = 0$ for all $\alpha \in \mathbb{N}^n$ for some $z_0 \in \Omega$, then $f \equiv 0$.*

Proof. The proof is the same as for the 1-dimensional case. $\square$

18.2 The maximum principle

Theorem 18.2 (maximum principle)**.** *Let $\Omega \subseteq \mathbb{C}^n$ be open and connected. If $f \in \mathrm{Hol}(\Omega)$ and $|f|$ assumes a local maximum in Ω, then f is constant.*

Proof. Let $z_0 \in \Omega$ be such that $|f(z_0)| \geq |f(z)|$ for all z in a neighborhood of z_0. Let $r > 0$ be such that $\{|z - z_0| < r\} \subseteq \Omega$, and consider $g_a(\tau) = f(z_0 + a\tau)$, where $a \in \mathbb{C}^n$ with $|a| = 1$ and $|\tau| < r$. Then $g_a \in \mathrm{Hol}(|\tau| < r)$, and $|g_a|$ has a local maximum at 0. So $g_a(\tau) = g_a(0)$ in $|\tau| < r$ by the maximum principle for $\mathbb{C}$. Since a is arbitrary, we get $f(z) = f(z_0)$ in $|z - z_0| < r$. By the previous corollary, $f = f(z_0)$ in Ω. □

18.3 Hartogs' lemma

We will prove the following theorem.

Theorem 18.3 (Hartogs' theorem on separately holomorphic functions)**.** *Let $\Omega \subseteq \mathbb{C}^n$ be open, and let $u : \Omega \to \mathbb{C}$ be separately holomorphic (holomorphic in each variable z_j, when the other variables are kept fixed). Then $u \in \mathrm{Hol}(\Omega)$.*

Remark 18.1. We do not even assume that u is measurable.

Remark 18.2. The corresponding result in the real domain is not true: for

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0), \end{cases}$$

$x \mapsto f(x, y)$ and $y \mapsto f(x, y)$ are real analytic, but f is not continuous at $(0, 0)$ (let alone differentiable).

Here is our starting point.

Proposition 18.1 (Hartogs' lemma)**.** *Let $\Omega \subseteq \mathbb{C}$ be open, and let (u_j) be subharmonic in Ω such that for all compact $K \subseteq \Omega$, there exists an M_K such that $u_j(z) \leq M_K$ for all $z \in K$ and $j = 1, 2, \ldots$. Assume that there is a $C < \infty$ such that for all $z \in \Omega$*

$$\limsup_{j \to \infty} u_j(z) \leq C.$$

Then for every compact set $K \subseteq \Omega$ and each $\varepsilon > 0$, there exists an N such that for all $j \geq N$,

$$u_j(z) \leq C + \varepsilon, \qquad z \in K.$$

Proof. Replacing Ω by a relatively compact domain containing K, we can assume that (u_j) is bounded above in Ω or even that $u_j \leq 0$ in Ω. Given compact $K \subseteq \Omega$, let $0 < r < \mathrm{dist}(K, \Omega^c)/3$ and recall the sub-mean value property:

$$u_j(z) \leq \frac{1}{\pi r^2} \iint_{|z-\zeta| \leq r} u_j(\zeta)\, d\lambda(\zeta), \qquad z \in K.$$

By Fatou's lemma,

$$\limsup_{j\to\infty} \iint_{|z-\zeta|\le r} u_j(\zeta)\,d\lambda(\zeta) \le \iint_{|z-\zeta|\le r} \limsup_{j\to\infty} u_j(\zeta)\,d\lambda(\zeta) \le C\pi r^2.$$

Thus, for all $z \in K$, there exists j_z such that if $j \ge j_z$, then

$$\iint_{|z-\zeta|\le r} u_j(\zeta)\,d\lambda(\zeta) \le \pi r^2 (C+\varepsilon/2).$$

We can assume here that $C+\varepsilon < 0$.

Let $|z-w| < \delta < r$. Then

$$u_j(w) \le \frac{1}{\pi(r+\delta)^2} \iint_{|\zeta-w|\le r+\delta} u_j(\zeta)\,d\lambda(\zeta).$$

Here, $\{\zeta : |\zeta - w| \le r+\delta\} \supseteq \{\zeta : |\zeta - z| \le r\}$. So

$$u_j(w) \le \frac{1}{\pi(r+\delta)^2} \underbrace{\iint_{|\zeta-z|\le r} u_j(\zeta)\,d\lambda(\zeta)}_{\le \pi r^2(C+\varepsilon/2)} \le \left(\frac{r}{r+\delta}\right)^2 (C+\varepsilon/2)$$

for $j \ge j_z$. Try to take $\delta = \mu r$ for $0 < \mu < 1$. The right hand side becomes

$$\frac{1}{(1+\mu)^2}(C+\varepsilon/2),$$

and we can take μ so this is just $C+\varepsilon$. So we can take

$$\mu = \underbrace{\left(\frac{C+\varepsilon/2}{C+\varepsilon}\right)^{1/2}}_{>1} - 1.$$

We can cover K by finitely many neighborhoods of the form $\{|z-w|<\delta\}$ for $z \in K$. $\square$

Next time, we will prove the following lemma on our road to Hartogs' theorem.

Lemma 18.1. *Let $\Omega \subseteq \mathbb{C}^n$ be open, and let u be separately holomorphic in Ω. If u is locally bounded in Ω, then $u \in C(\Omega)$ (so $u \in \mathrm{Hol}(\Omega)$).*

19 Hartogs' Theorem

19.1 Lemmas containing the argument

The goal is to prove the following theorem.

Theorem 19.1 (Hartogs)**.** *Let $\Omega \subseteq \mathbb{C}^n$ be open, and let $u : \Omega \to \mathbb{C}$ be separately holomorphic. Then $u \in \mathrm{Hol}(\Omega)$.*

We will break up the proof into a few lemmas.

Lemma 19.1. *Let $\Omega \subseteq \mathbb{C}^n$ be open, and let u be separately holomorphic in Ω. If u is locally bounded in Ω, then $u \in C(\Omega)$ (so $u \in \mathrm{Hol}(\Omega)$).*

Proof. Let D be a polydisc with $\overline{D} \subseteq \Omega$. Write $D = D_1 \times D'$, where D_1 is a disc in $\mathbb{C}$ and D' is a polydisc in $\mathbb{C}^{n-1}$. The function $z_1 \mapsto u(z_1, z') \in \mathrm{Hol}(D_1)$. By Cauchy's integral formula, $\partial_{z_1} u(z_1, z')$ is bounded when $z_1 \in D_1' \subseteq D_1$ (compactly contained) and $z' \in D'$. It follows that $\partial_{z_j} u$ is bounded on a relatively compact polydisc $\subseteq D$; in other words, $\partial_{z_j} u$ are locally bounded in Ω. Also, $\partial_{\overline{z}_j} = 0$ for all j.

It follows that u is continuous. If $a \in \Omega$ and $h \in \mathbb{C}^n \cong \mathbb{R}^{2n}$,

$$u(a+h) - u(a) = \sum_{j=1}^{2n} u(a + v_j) - u(a + v_{j-1}), \qquad v_j = (h_j, \dots, h_j, 0, \dots, 0).$$

Now use the mean value theorem. $\square$

Induction on n: Now assume that Hartogs' theorem is already known for functions of $< n$ complex variables.

Lemma 19.2. *Let $u : \Omega \to \mathbb{C}$ be separately holomorphic, and let $D = \prod_{j=1}^n D_j$ be a closed polydisc $\subseteq \Omega$ with $D^o \neq \varnothing$. Then there exist discs $D_j' \subseteq D_j$ for $1 \leq j \leq n-1$ with nonempty interior such that if $D_n' = D_n$, then u is bounded on $D' = \prod_{j=1}^n D_j'$.*

Proof. Let $E_M = \{z' \in \prod_{j=1}^{n-1} D_j : |u(z', z_n)| \leq M \ \forall z_n \in D_n\}$. E_M is closed: by the inductive hypothesis, $z' \mapsto u(z', z_n)$ is holomorphic in a neighborhood of $\prod_{j=1}^{n-1} D_j$ for each z_n and thus continuous; so

$$E_M = \bigcap_{z_n \in D_n} \left\{ z' \in \prod_{j=1}^{n-1} D_j : |u(z', z_n)| \leq M \right\}$$

is an intersection of closed sets. Also, $\bigcup_{M=1}^\infty E_M = \prod_{j=1}^{n-1} D_j$: $z_n \mapsto u(z', z_n)$ is holomorphic near D_n for all $z' \in \prod_{j=1}^{n-1}$ and is thus bounded on D_n: $|u(z', z_n)| \leq M$ for $z_n \in D_n$.

$\prod_{j=1}^{n-1} D_j$ is a complete metric space, so by Baire's theorem, so E_M has nonempty interior for some M. So E_M contains a polydisc $D' = \prod_{j=1}^{n-1} D_j'$ with nonempty interior such that if $D_n' = D_n$, u is bounded in $D' = \prod_{j=1}^n D_j' \subseteq D'$. $\square$

Lemma 19.3. *Let D be a polydisc $\{|z_j - z_j| < R : j = 1, \dots, n\}$. Let $u : D \to \mathbb{C}$ be holomorphic in $z' = (z_1, \dots, z_{n-1})$ for every fixed z_n, and assume that u is holomorphic and bounded in D' given by $|z_j - z_j^o| < r$ for all $1 \le j \le n-1$ for some $r > 0$ and $|z_n - z_n^o| < R$. Then $u \in \operatorname{Hol}(D)$.*

Proof. We may assume that $z^o = 0$. Take $0 < R_1 < R_2 < R$. Taylor expand $z' \mapsto u(z', z_n)$:

$$u(z', z_n) = \sum_{\alpha' \in \mathbb{N}^{n-1}} a_{\alpha'}(z_n)(z')^{\alpha'}, \qquad |z_j| < R, 1 \le j \le n-1, |z_n| < R.$$

We have that

$$a_{\alpha'}(z_n) = \frac{\partial^{\alpha'}(0, z_n)}{(\alpha')!}$$

is holomorphic in $|z_n| < R$. This series converges normally in $|z_j| < R$ for $1 \le j \le n-1$. So $a_{\alpha'}(z_n) R_2^{|\alpha'|} \to 0$ as $|\alpha'| \to \infty$ for each z_n. Now we have that $|u| \le M$ in D'. By Cauchy's estimates in z', we know that

$$|a_{\alpha'}(z_n)| \le \frac{M}{r^{|\alpha'|}} \qquad \forall \alpha'.$$

Consider the sequence of subharmonic (in $|z_n| < R$) functions

$$\varphi_{\alpha'}(z_n) = \frac{1}{|\alpha'|} \log |a_{\alpha'}(z_n)|, \qquad |\alpha'| = \alpha_1 + \cdots \alpha_{n-1}.$$

Our bound gives us that $\varphi_{\alpha'}$ is uniformly bounded above in $|z_n| < R$. Since $a_{\alpha'}(z_n) R_2^{|\alpha'|} \to 0$ as $|\alpha'| \to \infty$,

$$\limsup_{|\alpha'| \to \infty} \varphi_{\alpha'}(z_n) \le \log(1/R_2)$$

for all z_n. By Hartogs' lemma on subharmonic functions, if $|z_n| \le R_n$, then for any $\varepsilon > 0$,

$$\varphi_{\alpha'}(z_n) \le \log(1/R_2) + \varepsilon \le \log(1/R_1)$$

for large $|\alpha'|$. In other words, for large $|\alpha'|$ and $|z_n| \le R_2$,

$$|a_{\alpha'}(z_n)| R_1^{|\alpha_1|} \le 1.$$

The series $\sum_{\alpha' \in \mathbb{N}^{n-1}} a_{\alpha'}(z_n)(z')^{\alpha}$ converges absolutely for $|z_n| < R_2$ and $|z_j| < R_1$ (for all $1 \le j \le n-1$) and hence normally in D. So $u \in \operatorname{Hol}(D)$ as a limit of holomorphic functions (the partial sums). $\square$

19.2 Proof of the theorem from the lemmas

We can now prove Hartogs' theorem.

Proof. Let $z^0 \in \Omega$, and take a closed polydisc $\{|z_j - z_j^0| < 2R, 1 \leq j \leq n\}$. Apply the second lemma to the closed polydisc with $|z_j - z_j^0| \leq R$ for $1 \leq j \leq n-1$ and $|z_n - z_n^0| \leq 2R$. Then we get a polydisc of the form $|z_j - \zeta_j^0| < r$ for $1 \leq j \leq n-1$ and $|z_n - z_n^0| < R$ with $\{|z_j - \zeta_j^0| < r\} \subseteq \{|z_j - z_j^0| < R, 1 \leq j \leq n-1\}$ such that u is holomorphic and bounded there. In particular, $|z_j - z_j^0|$. In particular, $|\zeta_j^0 - z_j^0| < R$.

Consider the polydisc D given by $|z_j - \zeta_j^0| < R$ for $1 \leq j \leq n-1$ and $|z_n - z_n^0| < R$ (closure in Ω): in the polydisc, u is holomorphic in z' if z_n is fixed, and u is holomorphic and bounded in the polydisc $|z_j - \zeta_j^0| < r$ for $j = 1, \ldots, n$ and $|z_n - z_n^0| < R$. By the third lemma, u is holomorphic in D, which is a neighborhood of z_0. $\square$

20 Failure of the Riemann Mapping Theorem and Solving the $\overline{\partial}$-Equation

20.1 Failure of the Riemann mapping theorem in several complex variables

Theorem 20.1 (Poincaré)**.** *Let $D = \{z \in \mathbb{C} : |z| < 1\}$, and let $D^2 = D_z \times D_w \subseteq \mathbb{C}^2$ be the unit bidisc. There is no biholomorphic map $D^2 \to B_2 = \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 < 1\}$, the unit ball in $\mathbb{C}^2$.*

Remark 20.1. The Riemann mapping theorem does not hold for domains in $\mathbb{C}^n$ for $n > 1$.

Remark 20.2. Intuition: ∂D^2 contains non-constant analytic discs (holomorphic $f : D \to \partial D^2$), while ∂B_2 does not.

Proof. Assume that there exists a biholomorphic map $f : D^2 \to B_2$. Write $f(z, w) = (f^1(z, w), f^2(z, w))$. Let $w_0 \in \partial D_w$, and let $w_n \in D$ be such that $w_n \to w_0$. Then for any $z \in D$, $(z, w_n) \to (z, w_0) \in \partial D^2$. Then $|f(z, w_n)| \to 1$ (here, we only use that f is **proper**: for any compact $K \subseteq B_2$, $f^{-1}(K)$ is compact).

On the other hand, we have $g_n(z) := f(z, w_n) \in \mathrm{Hol}(D, \mathbb{C}^2)$ with $|g_n(z)| \leq 1$. By normal families, passing to a subsequence, we get $g_n \to g \in \mathrm{Hol}(D, \mathbb{C}^2)$ locally uniformly. We have $|g(z)| = 1$ for all $z \in D$.

We claim that $g(z)$ is constant. Write $g(z) = (g^1(z), g^2(z))$, where

$$|g^1(z)|^2 + |g^2(z)|^2 = 1 \qquad z \in D.$$

Apply ∂_z:

$$(\partial_z g^1)\overline{g^1} + (\partial_z g^2)\overline{g^2} = 0.$$

Apply $\partial_{\overline{z}}$:

$$|\partial_z g^1|^2 + |\partial_z g^2|^2 = 0.$$

So $\partial_z g^i = 0$, and we get the claim.

Thus, $f(z, w_n)$ converges to a constant so that $f'_z(z, w_n) \to 0$. Let $z = z_0 \in D$ be fixed, and consider $h(w) = f'_z(z_0, w) = (h^1(w), h^2(w)) \in \mathrm{Hol}(D, \mathbb{C}^2)$. Write by Cauchy's integral formula:

$$h^j(w) = \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{f^j(\zeta, w)}{(\zeta - z_0)^2}\, d\zeta, \qquad |z_0| < r < 1.$$

h is bounded in D, so the radial limits $\lim_{r \to 1} h(rw_0)$ exist for almost every $w_0 \in \partial D$. We have that $h(w_n) \to 0$ if $w_n \to w_0 \in \partial D$. It follows that $\lim_{r \to 1} h(rw_0) = 0$ for almost every $w_0 \in \partial D$, and by the uniqueness theorem, $h(w) \equiv 0$ for $|w| < 1$. We get that $f'_z(z, w) = 0$ for all $(z, w) \in D^2$, so $f = f(w)$. Replacing the role of z and w, we get that f is constant. $\square$

20.2 Solving the $\overline{\partial}$-equation with compactly supported right hand side

Recall that if $\varphi \in C_0^k(\mathbb{C})$ with $k \geq 1$ and we set

$$u(z) = -\frac{1}{\pi} \iint \frac{\varphi(\zeta)}{\zeta - z} L(d\zeta),$$

then $u \in C^k(\mathbb{C})$, and $\frac{\partial u}{\partial \overline{z}} = \varphi$.

Remark 20.3. In general, the equation $\frac{\partial u}{\partial \overline{z}} = \varphi$ has no solutions with compact support.

In $\mathbb{C}^n$, when $n > 1$, the $\overline{\partial}$-equation is a system:

$$\frac{\partial u}{\partial \overline{z}_j} = f_j, \qquad 1 \leq j \leq n.$$

This is an overdetermined system, which cannot be solved unless the right hand side satisfies the compatibility conditions

$$\frac{\partial f_j}{\partial \overline{z}_k} = \frac{\partial f_k}{\partial \overline{z}_j} \qquad 1 \leq j, k \leq n.$$

Remark 20.4. If we view $\overline{\partial} u = \sum_{j=1}^n \frac{\partial u}{\partial \overline{z}_j} \, d\overline{z}_j$ as a 1-form and introduce the 1-form $f = \sum_{j=1}^n f_j \, d\overline{z}_j$, then the system becomes

$$\overline{\partial} u = f.$$

If we define the 2-form $\overline{\partial} f = \sum_{j=1}^n \overline{\partial} f_j \wedge d\overline{z}$, then the compatibility conditions become $\overline{\partial} f = 0$:

$$\overline{\partial} f = \sum_{j=1}^n \left(\sum_{k=1}^n \frac{\partial f_j}{\partial \overline{z}_k} \, d\overline{z}_k \right) \wedge dz_j = \sum_{j<k} \left(\frac{\partial f_j}{\partial \overline{z}_k} - \frac{\partial f_k}{\partial z_j} \right) d\overline{z}_k \wedge d\overline{z}_j$$

Theorem 20.2. *Let $f_j \in C_0^k(\mathbb{C}^n)$ for $1 \leq j \leq n$ and $n > 1$ be such that $\overline{\partial} f = 0$. Then the equation $\overline{\partial} u = f$ has a solution $u \in C_0^k(\mathbb{C}^n)$.*

Remark 20.5. Such a solution is unique: if $u, \tilde{u} \in C_0^k(\mathbb{C}^n)$, then $\overline{\partial}(u - \tilde{u}) = 0$. So $u - \tilde{u} \in \mathrm{Hol}(\mathbb{C}^n)$ with compact support. So $u = \tilde{u}$.

Proof. Consider $\frac{\partial u}{\partial \overline{z}_j}$ for $1 \leq j \leq n$. Define

$$u(z) = -\frac{1}{\pi} \iint_{\mathbb{C}} \frac{f_1(\zeta_1, z_2, \ldots, z_n)}{\zeta_1 - z_1} L(d\zeta_1) = -\frac{1}{\pi} \iint \frac{f_1(\zeta_1 + z_1, z_2, \ldots, z_n)}{\zeta_1} L(d\zeta_1).$$

Then $u \in C^k(\mathbb{C}^n)$, and $\frac{\partial u}{\partial \overline{z}_1} = f_1$. $\square$

We will continue the proof next time.

21 The $\overline{\partial}$-Equation, the Hartogs Extension Theorem, and Regularization of Subharmonic Functions

21.1 Compactly supported solutions of the $\overline{\partial}$-equation

Theorem 21.1. *Let $f_j \in C_0^k(\mathbb{C}^n)$ for $1 \leq j \leq n$ and $n > 1$ be such that $\overline{\partial} f = 0$. Then the equation $\overline{\partial} u = f$ has a unique solution $u \in C_0^k(\mathbb{C}^n)$.*

Proof. Consider $\frac{\partial u}{\partial \overline{z}_j}$ for $1 \leq j \leq n$. Define

$$u(z) = -\frac{1}{\pi} \iint_{\mathbb{C}} \frac{f_1(\zeta_1, z_2, \dots, z_n)}{\zeta_1 - z_1} L(d\zeta_1).$$

Then $u \in C^k(\mathbb{C}^n)$, and $\frac{\partial u}{\partial \overline{z}_1} = f_1$. When $j > 1$, we have by the compatibility conditions that

$$\frac{\partial u}{\partial \overline{z}_j} = -\frac{1}{\pi} \iint \frac{\frac{\partial f_1}{\partial \overline{z}_j}(\zeta_1, z_2, \dots, z_n)}{\zeta - z_1} L(d\zeta_1) = \frac{1}{\pi} \iint \frac{\frac{\partial f_1}{\partial \overline{z}_1}(\zeta_1, z_2, \dots, z_n)}{\zeta - z_1} L(d\zeta_1) = f_j(z),$$

using Cauchy's integral formula.

We claim that if $n > 1$, then u is compactly supported: If $|z_1| + \cdots + |z_n|$ is large enough, then $u(z) = 0$. On the other hand, $\overline{\partial} u = 0$ on $\mathbb{C}^n \setminus K$, where $K = \bigcup_{i=1}^n \operatorname{supp}(f_i)$ is compact. $u \in \operatorname{Hol}(\mathbb{C}^n \setminus K)$, and if Ω is the unbounded component, then, as $u(z) = 0$ on some open set in Ω, $u = 0$ in Ω by the uniqueness of analytic continuation. So $\operatorname{supp}(u) \subseteq K \cup \bigcup \mathcal{M}$, where M is a bounded component of $\mathbb{C}^n \setminus K$. This is bounded, so $u \in C_0^k(\mathbb{C}^n)$. $\square$

21.2 The Hartogs extension theorem

Theorem 21.2 (Hartogs extension theorem)**.** *Let $|Omega \subseteq \mathbb{C}^n$ be open with $n > 1$, and let $K \subseteq \Omega$ be compact with $\Omega \setminus K$. Let $u \in \operatorname{Hol}(\Omega \setminus K)$. Then there exists a $U \in \operatorname{Hol}(\Omega)$ such that $U = u$ in $\Omega \setminus K$.*

Proof. Let $\varphi \in C_0^\infty(\Omega)$ such that $\varphi = 1$ in a neighborhood of K. Then let $u_0 = (1 - \varphi) u \in C^\infty(\Omega)$. We shall construct a holomorphic extension U of u such that $U = u_0 - v$, where we need $v \in C^\infty(\Omega)$ and $\overline{\partial} U = 0$. We need

$$\begin{aligned}
0 &= \overline{\partial} U \\
&= \overline{\partial} u - \overline{\partial} v \\
&= \overline{\partial}((1 - \varphi) u) - \overline{\partial} v \\
&= (\overline{\partial}(1 - \varphi)) u - \partial \overline{v} \\
&= -(\overline{\partial} \varphi) u + \overline{\partial} v
\end{aligned}$$

with compact support $\subseteq \Omega$, away from K. Here, we have used that $u \in \mathrm{Hol}(\Omega \setminus K)$. We have that $(\overline{\partial}\varphi)u \in C_0^\infty(\mathbb{C}^n; \mathbb{C}^n)$. Solve:

$$\overline{\partial} v = -(\overline{\partial}\varphi) u.$$

The compatibility conditions are satisfied:

$$\partial_{\overline{z}_k}\left(\frac{\partial \varphi}{\partial \overline{z}_j} u\right) = \partial_{\overline{z}_j}\left(\frac{\partial \varphi}{\partial \overline{z}_k} u\right) \qquad \forall j, k.$$

So there exists a $v \in C_0^\infty(\mathbb{C}^n)$ solving this, and $\mathrm{supp}(\overline{\partial} v) \subseteq \mathrm{supp}(\varphi)$. So $v = 0$ on the unbounded component O of $\mathbb{C}^n \setminus \mathrm{supp}(\varphi)$. We get $U - u_0 - v = (1-\varphi)u - v \in \mathrm{Hol}(\Omega)$, and $U = u$ on $O \cap (\Omega \setminus \mathrm{supp}\,\varphi))$, which is an open subset of $\Omega \setminus K$. This is nonempty because $\partial O \subseteq \mathrm{supp}(\varphi)$, so since $\Omega \setminus K$ is connected, $U = u$ in $\Omega \setminus K$. □

The following special case is of note:

Corollary 21.1. *Let $f \in \mathrm{Hol}(\mathbb{C}^n)$ with $n > 1$. Then f cannot have an isolated zero.*

Proof. If $f(0) = 0$ and $f \neq 0$ on $0 < |z| < R$, then apply the Hartogs extension theorem to $K = \{0\}$ and $\Omega = \{|z| < R\}$.Then $h = 1/f \in \mathrm{Hol}(\Omega \setminus K)$, os there exists a extension $U \in \mathrm{Hol}(|z| < R)$. Then $fU = 1$, which is a contradiction. □

21.3 Regularization of subharmonic functions

Let $\Omega \subseteq \mathbb{C}$ be open and connected. Let $u \in \mathrm{SH}(\Omega)$ with $u \not\equiv -\infty$. Then $u \in L^1_{\mathrm{loc}}(\Omega)$. Let $0 \leq \varphi \in C_0^\infty(\mathbb{C})$ be such that $\mathrm{supp}(\varphi) \subseteq \{|z| < 1\}$ and $\int \varphi(z)\, L(dz) = 1$, where φ depends only on $|z|$.

Remark 21.1. We can take

$$\varphi(z) = Ch(1 - |z|^2), \qquad h(t) = \begin{cases} e^{-1/t} & t > 0 \\ 0 & t \leq 0. \end{cases}$$

You can check that $h^{(j)}(0) = 0$ for all j, so $h \in C^\infty(\mathbb{R})$.

Define

$$u_\varepsilon = u * \varphi_\varepsilon, \qquad \varphi_\varepsilon(z) = \frac{1}{\varepsilon^2}\varphi\left(\frac{z}{\varepsilon}\right),$$

so

$$u_\varepsilon(z) = \int u(z - \zeta)\varphi_\varepsilon(\zeta)\, L(d\zeta), \qquad z \in \Omega_\varepsilon = \{z \in \Omega : \mathrm{dist}(z, \Omega^c) > \varepsilon\}.$$

Proposition 21.1. *$u_\varepsilon \in (C^\infty \cap \mathrm{SH})(\Omega_\varepsilon)$, and $u_\varepsilon \downarrow u$ as $\varepsilon \downarrow 0$.*

Proof. We have

$$u_\varepsilon(z) = \frac{1}{\varepsilon^2} \int u(\zeta)\varphi\left(\frac{z-\zeta}{\varepsilon}\right) L(d\zeta) \in C^\infty(\Omega_\varepsilon).$$

Check the sub-mean value inequality: First write

$$u_\varepsilon(z) = \int u(z-\varepsilon\zeta)\varphi(\zeta)\, L(d\zeta).$$

If $z \in \Omega_\varepsilon$ and r is small, then since u is subharmonic,

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} u_\varepsilon(z+re^{it})\,dt &= \frac{1}{2\pi}\int_0^{2\pi}\int u(z+re^{it}-\varepsilon\zeta)\varphi(\zeta)\,L(d\zeta)\,dt \\
&\geq \int u(z-\varepsilon\zeta)\varphi(\zeta)\,L(d\zeta) \\
&= u_\varepsilon(z).
\end{aligned}$$

To show that $u_\varepsilon(z) \geq u(z)$, we have

$$\begin{aligned}
u_\varepsilon(z) &= \int u(z+\varepsilon\zeta)\varphi(\zeta)\,L(d\zeta) \\
&= \int_0^\infty \underbrace{\left(\int_0^{2\pi} u(z+\varepsilon re^{it})\,dt\right)}_{\geq 2\pi u(z)} \varphi(r) r\,dr \\
&\geq \underbrace{\left(2\pi\int_0^\infty \varphi(r) r\,dr\right)}_{=1} u(z).
\end{aligned}$$

□

We will finish the proof next time.

22 Regularization of Subharmonic Functions and L^2 Estimates for the $\overline{\partial}$ Operator

22.1 Regularization of subharmonic functions

Let $u \in \mathrm{SH}(\Omega)$ be $u \not\equiv -\infty$. Let $0 \leq \varphi \in C_0^\infty(\mathbb{C})$ be such that $\varphi = 0$ for $|z| \geq 1$, φ is radially symmetric, and $\int \varphi = 1$. Define

$$u_\varepsilon = u * \varphi_\varepsilon = \int u(z-\zeta)\varphi_\varepsilon(\zeta)\, L(d\zeta), \qquad \varphi_\varepsilon(z) = \frac{1}{\varepsilon^2}\varphi\left(\frac{z}{\varepsilon}\right),$$

and let $\Omega_\varepsilon = \{z \in \Omega : \mathrm{dist}(z, \Omega^c) > \varepsilon\}$,

Proposition 22.1. *$u_\varepsilon \in (C^\infty \cap \mathrm{SH})(\Omega_\varepsilon)$, and $u_\varepsilon \downarrow u$ as $\varepsilon \downarrow 0$.*

Proof. We have already shown the first statement, and we have shown that $u_\varepsilon \geq 0$ for all $\varepsilon > 0$.

We want to check that $u_\varepsilon \downarrow u$ as $\varepsilon \downarrow 0$. As φ is radially symmetric, we have

$$u_\varepsilon(z) = \int \varphi(r) r \underbrace{\left(\int_0^{2\pi} u(z + \varepsilon r e^{it})\, dt\right)}_{\text{increasing with } \varepsilon} dr.$$

We get that $\lim_{\varepsilon\to 0} u_\varepsilon \in \mathrm{SH}(\Omega)$ and is $\geq u$. On the other hand, by Fatou's lemma,

$$\limsup_{\varepsilon\to 0} \int u(z+\varepsilon\zeta)\varphi(\zeta)\, L(d\zeta) \leq \int \limsup_{\varepsilon\to 0} u(z+\varepsilon\zeta)\varphi(\zeta)\, L(d\zeta) \leq u(z)$$

by the upper semicontinuity of u. So $u_\varepsilon \downarrow u$. ☐

Remark 22.1. Regularization arguments show the following: if $u \in \mathrm{SH}(\Omega)$, where $u \not\equiv -\infty$ and Ω is connected, then

$$\int u \Delta\varphi\, L(ds) \geq 0 \qquad \forall 0 \leq \varphi \in C_0^\infty(\Omega).$$

Conversely, assume that $U \in L^1_{\mathrm{loc}}(\Omega)$ such that

$$\int U \Delta\varphi\, L(d\zeta) \geq 0.$$

Then there exists a unique $u \in \mathrm{SH}(\Omega)$ such that $u = U$ a.e.

22.2 L^2 estimates for the $\overline{\partial}$ operator

Let $\Omega \subseteq \mathbb{C}$ be open. Consider the Cauchy-Riemann equation

$$\frac{\partial u}{\partial \overline{z}} = f.$$

Recall thatif $f \in C^\infty(\Omega)$, there exists some $u \in C^\infty(\Omega)$ solving this equation. We want to solve the equation with $f \in L^2_{\text{loc}}(\Omega)$ and get *estimates* for the solution.

Definition 22.1. Let $f \in L^2_{\text{loc}}(\Omega)$. We say that $u \in L^2_{\text{loc}}$ is a **solution in the weak sense** of the Cauchy-Riemann equation if for all $\eta \in C_0^\infty(\Omega)$,

$$-\int u \partial_{\overline{z}} \beta \, L(dz) = \int f \beta \, L(dz).$$

Theorem 22.1 (Hörmander[10])**.** *Let $\Omega \subseteq \mathbb{C}$ be open, and let $\varphi \in C^\infty(\Omega)$ be strictly subharmonic: $\Delta\varphi > 0$ in Ω. Then, for any $f \in L^2_{\text{loc}}(\Omega)$ such that*

$$\int \frac{|f|^2}{\varphi''_{z,\overline{z}}} e^{-\varphi} \, L(dz) < \infty,$$

there exists a weak solution $u \in L^2_{\text{loc}}(\Omega)$ to $\frac{\partial u}{\partial \overline{z}} = f$ such that

$$\int_\Omega |u|^2 e^{-\varphi} \, L(dz) \leq \int_\Omega \frac{|f|^2}{\varphi''_{z,\overline{z}}} e^{-\varphi} \, L(dz).$$

Proof. We shall work in the Hilbert space

$$L^2_\varphi = L^2(\Omega, e^{-\varphi}) = \left\{ f : \Omega \to \mathbb{C} \text{ measurable} \mid \|f\|_{L^2_\varphi} := \int |f| e^{-\varphi} \, L(dz) < \infty \right\}.$$

Consider the linear operator $T : L^2_\varphi \to L^2_\varphi$ given by $Tu = \frac{\partial u}{\partial \overline{z}}$ equipped with the domain

$$D(T) = \left\{ u \in L^2_\varphi : \exists f \in L^2_\varphi \text{ s.t. } f = \frac{\partial u}{\partial \overline{z}} \text{ weakly: } -\int u \partial_{\overline{z}} \beta = \int f \beta \; \forall \beta \in C_0^\infty(\Omega) \right\}.$$

Then $D(T)$ is dense in L^2_φ, and $Tu = f$.

We have the adjoint $T^* =: \overline{\partial}^*_\varphi$ of T:

$$\langle \overline{\partial}, \beta \rangle_{L^2_\varphi} = \langle u, \overline{\partial}^*_\varphi \beta \rangle_{L^2_\varphi} \qquad \forall u \in D(T), \beta \in C_0^\infty(\Omega).$$

[10]This result, unlike the other results we have been proving, is fairly recent. It was proven in 1965.

Compute $\overline{\partial}^*_\varphi$:

$$\langle \partial \overline{u}, \beta\rangle_{L^2_\varphi} = \int \overline{\partial} u \underbrace{\overline{\beta} e^{-\varphi}}_{\in C_0^\infty} L(dz) = -\int u \partial_{\overline{z}}(\overline{\beta} e^{-\varphi})\, L(dz) = \int u \overline{\overline{\partial}^*_\varphi \beta} e^{-\varphi}\, L(dz).$$

We get that

$$\overline{\partial}^*_\varphi \beta = -e^{\varphi}\partial_z(\beta e^{-\varphi}) = -\partial_z \beta + (\partial_z \varphi)\beta.$$

The idea is that to get a solvability result for $\overline{\partial}$ acting on L^2_φ, we need an a priori estimate for $\overline{\partial}^*_\varphi$. □

Before we continue with the proof, we need the following proposition:

Proposition 22.2. *Let $f \in L^2_{\mathrm{loc}}(\Omega)$, and let $C > 0$ be constant. Then there exists a $u \in L^2_{\mathrm{loc}}(\Omega)$ such that $\overline{\partial} u = f$ and $\int |u|^2 e^{-\varphi}\, L(dz) \le C$ if and only if*

$$\left| \int f \overline{\beta} e^{-\varphi}\, L(dz) \right| \le C \int |\overline{\partial}^*_\varphi \beta|^2 e^{-\varphi}\, L(dz) \qquad \forall \beta \in C_0^\infty(\Omega).$$

Proof. ($\implies$): We have by Cauchy-Schwarz that

$$\left| \int f \overline{\beta} e^{-\varphi}\, L(dz) \right| = \left| \int \overline{\partial} u \overline{\beta} e^{-\varphi}\, L(dz) \right| = |\langle u, \overline{\partial}^*_\varphi \beta\rangle_{L^2_\varphi}| \le C^{1/2} \|\overline{\partial}^*_\varphi \beta\|_{L^2_\varphi}.$$

($\impliedby$): Assume that the bound holds. The linear functional

$$F(\overline{\partial}^*_\varphi \beta) = \overline{\int f \overline{\beta} e^{-\varphi}\, L(dz)}.$$

is well-defined on $\overline{\partial}^*_\varphi C_0^\infty(\Omega) \subseteq L^2_\varphi$, and $|F(\overline{\partial}^*_\varphi \beta)| \le C^{1/2} \|\overline{\partial}^*_\varphi \beta\|_{L^2_\varphi}$. So its norm is $\le C^{1/2}$. By the Hahn-Banach theorem, F extends to all of L^2_φ. So there is a $u \in L^2_\varphi$ representing the linear functional F. □

23 Hömander's Theorem for Solving the $\overline{\partial}$-Equation in One Variable

23.1 Completion of the proof of Hömander's theorem

We want to solve $\overline{\partial}u = f$ on $\Omega \subseteq \mathbb{C}$. Last time, we were proving the following observation:

Proposition 23.1. *Let $f \in L^2_{\text{loc}}(\Omega)$, and let $C > 0$ be constant. Then there exists a $u \in L^2_{\text{loc}}(\Omega)$ such that $\overline{\partial}u = f$ and $\int |u|^2 e^{-\varphi}\, L(dz) \leq C$ if and only if*

$$\left| \int f\overline{\beta}e^{-\varphi}\, L(dz) \right| \leq C \int |\overline{\partial}^*_\varphi \beta|^2 e^{-\varphi}\, L(dz) \qquad \forall \beta \in C_0^\infty(\Omega).$$

Proof. ($\Longleftarrow$): Consider the linear map $F : \overline{\partial}^*_\varphi C_0^\infty(\Omega) \to \mathbb{C}$ given by$F(\overline{\partial}^*_\varphi \beta) = \int f\overline{\beta}e^{-\varphi}$. Then

$$|F(\overline{\partial}^*_\varphi \beta)| \leq C^{1/2} \|\overline{\partial}^*_\varphi \beta\|_{L^2},$$

By the Hahn-Banach theorem, F extends to a linear continuous map on L^2_φ with the norm $\leq C^{1.2}$. Thus, there exists a $u \in L^2_\varphi$ with $\|u\|_{L^2_\varphi} \leq C^{1/2}$ such that $F(g) = \langle g, h \rangle_{L^2_\varphi}$ for all $g \in L^2\varphi$. In particular, if $g = \overline{\partial}^*_\varphi \beta$,

$$\overline{\int f\overline{\beta}e^{-\varphi}} = \langle \overline{\partial}^*_\varphi \beta, u \rangle_{L^2_\varphi} \qquad \forall \beta \in C_0^\infty.$$

We get

$$\int f\,\overline{\beta}e^{-\varphi} = -\int u\partial_{\overline{z}}(e^{-\varphi\overline{\beta}}).$$

for all β. So we get that $\overline{\partial}u = f$ weakly. $\square$

We can now complete the proof of Hörmander's theorem.

Theorem 23.1 (Hörmander[11]). *Let $\Omega \subseteq \mathbb{C}$ be open, and let $\varphi \in C^\infty(\Omega)$ be strictly subharmonic: $\Delta\varphi > 0$ in Ω. Then, for any $f \in L^2_{\text{loc}}(\Omega)$ such that*

$$\int \frac{|f|^2}{\varphi''_{z,\overline{z}}} e^{-\varphi}\, L(dz) < \infty,$$

there exists a weak solution $u \in L^2_{\text{loc}}(\Omega)$ to $\frac{\partial u}{\partial \overline{z}} = f$ such that

$$\int_\Omega |u|^2 e^{-\varphi}\, L(dz) \leq \int_\Omega \frac{|f|^2}{\varphi''_{z,\overline{z}}} e^{-\varphi}\, L(dz).$$

[11] This result, unlike the other results we have been proving, is fairly recent. It was proven in 1965.

Proof. We need to show that

$$\left|\int f\overline{\beta}e^{-\varphi}\right|^2 \leq C\|\overline{\partial}^*_\varphi\beta\|^2_{L^2_\varphi} \qquad \forall \beta \in C_0^\infty.$$

We need a lower bound for $\|\overline{\partial}^*_\varphi\beta\|^2_{L^2_\varphi}$:

In general, let H be a Hilbert space, and let $T \in \mathcal{L}(H,H)$. Then

$$\begin{aligned}\|T^*x\|^2 \geq \|T^*x\|^2 - \|Tx\|^2 &= \langle T^*x, T^*x\rangle - \langle Tx, Tx\rangle \\ &= \langle TT^*x, x\rangle - \langle T^*Tx, x\rangle \\ &= \langle [T,T^*]x, x\rangle,\end{aligned}$$

where $[T,T^*] = TT^* - T^*T$ is the commutator of T, T^*. In our case, $H = L^2_\varphi$, $T = \overline{\partial}$, and $T^* = \overline{\partial}^*_\varphi = -\partial_z + \partial_z\varphi$. So The commutator is

$$[\overline{\partial}, \overline{\partial}^*_\varphi] = [\overline{\partial}, -\partial + \partial\varphi] = \cancelto{0}{[\overline{\partial}, \partial]} + [\overline{\partial}, \partial\varphi].$$

Compute for $\beta \in C_0^\infty$:

$$[\overline{\partial}, \partial\varphi]\beta = \overline{\partial}(\partial\varphi\beta) - \partial\varphi\overline{\partial}\beta = \underbrace{(\overline{\partial}\partial\varphi)}_{\Delta\varphi/4>0}\beta.$$

We get that

$$\|\overline{\partial}^*_\varphi\beta\|^2_{L^2_\varphi} \geq \frac{1}{4}\int \Delta\varphi|\beta|^2e^{-\varphi} \qquad \forall \beta \in C_0^\infty(\Omega).$$

It follows by Cuachy-Schwarz in L^2_φ that

$$\left|\int f\overline{\beta}e^{-\varphi}\right| \leq \left(\int \frac{|f|^2}{\Delta\varphi}e^{-\varphi}\right)\underbrace{\left(\int \Delta\varphi|\beta|^2e^{-\varphi}\right)}_{\leq 4\|\overline{\partial}^*_\varphi\beta\|^2_{L^2_\varphi}}.$$

Finally, we get that there exists some $u \in L^2_\varphi$ such that $\overline{\partial}u = f$ and

$$\|u\|^2_{L^2_\varphi} \leq 4\int \frac{|f|^2}{\Delta\varphi}e^{-\varphi}.$$

□

Remark 23.1. $\overline{\partial}^*_\varphi C_0^\infty(\Omega) \subseteq L^2_\varphi$: we obtain $u \in \overline{\overline{\partial}^*_\varphi C_0^\infty(\Omega)}$ such that if $h \in \ker(\overline{\partial}) \cap L^2_\varphi$ (i.e. h is holomorphic), then

$$0 = \left\langle \overline{\partial}h, \beta\right\rangle = \langle h, \overline{\partial}^*_\varphi\beta\rangle_{L^2_\varphi} \qquad \forall \beta \in C_0^\infty.$$

So $u \perp \ker(\overline{\partial}) \cap L^2_\varphi$. Thus, we have found a solution of $\overline{\partial}u = f$ of minimal norm in L^2_φ.

23.2 Weakening the assumptions of Hörmander's theorem

Assume that $\varphi \in C^\infty(\Omega)$ is just subharmonic: $\Delta\varphi \geq 0$. Apply Hörmander's theorem to

$$\psi(z) = \varphi(z) + a\log(1+|z|^2), \qquad a > 0.$$

We can estimate (setting $r = |z|$):

$$\Delta\psi(z) \geq a \underbrace{\Delta \log(1+|z|^2)}_{=(\partial_r^2 + \frac{1}{r}\partial_r)(\log(1+r^2)))} = \frac{4}{(1+r^2)^2}.$$

We get that $\overline{\partial}u = f$ has a solution $u \in L^2_{\mathrm{loc}}$ such that

$$a\int_\Omega |u|^2 e^{-\varphi}(1+|z|^2)^{-a} \leq \int |f|^2 e^{-\varphi}(1+|z|^2)^{2-a}$$

for all subharmonic $\varphi \in C^\infty$.

It turns out that the same estimate is valid for any subharmonic function, not just ones in C^∞.

Theorem 23.2. *Let $\Omega \subseteq \mathbb{C}$ be open and connected, and let $\varphi \in \mathrm{SH}(\Omega)$ with $\varphi \not\equiv -\infty$. Let $a > 0$, and assume that $f \in L^2_{\mathrm{loc}}$ is such that*

$$\int |f|^2 e^{-\varphi}(1+|z|^2)^{2-a} < \infty.$$

Then there exists a u solving $\overline{\partial}u = f$ such that

$$a\int_\Omega |u|^2 e^{-\varphi}(1+|z|^2)^{-a} \leq \int |f|^2 e^{-\varphi}(1+|z|^2)^{2-a}.$$

We will prove this next time.

Remark 23.2. Let $f \in L^2_{\mathrm{loc}}(\Omega)$. Then there is a $u \in L^2_{\mathrm{loc}}(\Omega)$ solving $\overline{\partial}u = f$: there exists a $\varphi \in C(\Omega) \cap \mathrm{SH}(\Omega)$ such that $f \in L^2(\Omega, e^\varphi)$ (that is, $\int |f|^2 e^{-\varphi} < \infty$: for $\Omega \neq \mathbb{C}$, take

$$\varphi_0(z) = -\log(\mathrm{dist}(z, \Omega^c)),$$

which is subharmonic in Ω with the property that $\varphi_0(z) \to \infty$ as $z \to \partial\Omega$. Composing φ_0 with a suitable convex increasing function, we get φ such that the bound holds.

24 General Hörmander's Theorem and Application to Interpolation by Holomorphic Functions

24.1 Hörmander's theorem for arbitrary subharmonic functions

Theorem 24.1. *Let $\Omega \subseteq \mathbb{C}$ be open and connected, and let $\varphi \in \mathrm{SH}(\Omega)$ with $\varphi \not\equiv -\infty$. Let $a > 0$, and assume that $f \in L^2_{\mathrm{loc}}$ is such that*

$$\int |f|^2 e^{-\varphi}(1+|z|^2)^{2-a} < \infty.$$

Then there exists a u solving $\overline{\partial} u = f$ such that

$$a \int_\Omega |u|^2 e^{-\varphi}(1+|z|^2)^{-a} \leq \int |f|^2 e^{-\varphi}(1+|z|^2)^{2-a}.$$

Proof. This estimate has been proved if $\varphi \in C^\infty$. In general, let $\Omega_j \subseteq \Omega$ be open, relatively compact, and increasing to Ω, and let $\varphi_j \in C^\infty(\Omega_j) \cap \mathrm{SH}(\Omega_j)$ such that $\varphi_j \downarrow \varphi$. Then

$$\int_\Omega |f|^2 e^{-\varphi_j}(1+|z|^2)^{2-a} \leq \int_\Omega |f|^2 e^{-\varphi}(1+|z|^2)^{2-a} \leq C \qquad \forall j.$$

We get that there exists some u_j solving $\overline{\partial} u_j = f$ in Ω_j such that

$$\int_{\Omega_j} |u_j|^2 e^{-\varphi_j}(1+|z|^2)^{-a} \leq C, \qquad j = 1, 2, \dots.$$

Let j be fixed, and consider $(u_j)_{j=k}^\infty$ on Ω_k:

$$\int_{\Omega_k} |u_j|^2 e^{-\varphi_k}(1+|z|^2)^{-a} \leq \int_{\Omega_j} |u_j|^2 e^{-\varphi_j}(1+|z|^2)^{-a} \leq C.$$

So $(u_j)_{j=k}^\infty$ is bounded in $L^2(\Omega_k, e^{-\varphi_k})$.

Extracting a weakly convergent subsequence and using a diagonal argument, we get a subsequence (u_{j_ν}) and $u \in L^2_{\mathrm{loc}}(\Omega)$ such that $u_{j_\nu} \to u$ weakly in $L^2(\Omega_k, e^{-\varphi_k})$ for all k. Then $\overline{\partial} u = f$ in Ω: for any $\beta \in C_0^\infty(\Omega_k)$, $\int u_{j_\nu}\beta \to \int u\beta$, so $\overline{\partial} u_{j_\nu} = f$ on Ω_k for large ν. We have $-\int u_{j_\nu}\overline{\partial}\beta = \int f\beta$ and thus $\overline{\partial} u = f$ on Ω_K.

To get the bound for u, recall that if H is a Hilbert space and $x_j \to x$ weakly in H, then $\|x\| \leq \liminf_j \|x_j\|$. We get that for any k,

$$\begin{aligned} a \int_{\Omega_k} |u|^2 e^{-\varphi_k}(1+|z|^2)^{-a} &\leq \liminf_{\nu\to\infty} \int_{\Omega_k} |u_{j_\nu}|^2 e^{-\varphi_k}(1+|z|^2)^{-a} \\ &\leq \int_\Omega |f|^2 e^{-\varphi}(1+|z|^2)^{2-a}. \end{aligned}$$

Let $k \to \infty$ and use the monotone convergence theorem to get

$$\int_\Omega |u|^2 e^{-\varphi}(1+|z|^2)^{-a} \leq \int_\Omega |f|^2 e^{-\varphi}(1+|z|^2)^{2-a}. \qquad \square$$

24.2 Application: Interpolation by holomorphic functions

Here is an application of Hörmander's theorem.

Proposition 24.1. *Let $(b_k)_{k=-\infty}^{\infty}$ be a bounded sequence in $\mathbb{C}$. There exists an $h \in \mathrm{Hol}(\mathbb{C})$ with suitable growth properties such that $h(k) = b_k$ for every $k \in \mathbb{Z}$.*

Proof. Let us first find a C^∞ solution: let $\psi \in C_0^\infty(\mathbb{C})$ be such that

$$\psi(z) = \begin{cases} 1 & |z| \le 1/4 \\ 0 & |z| \ge 1/3. \end{cases}$$

Then $g(z) = \sum_{k \in \mathbb{Z}} b_k \psi(z-k)$ is locally finite and solves the problem. We have $g \in (C^\infty \cap L^\infty)(\mathbb{C})$. Try to construct $h \in \mathrm{Hol}(\mathbb{C})$ in the form $h = g - u$, where $0 = \overline{\partial} h = \overline{\partial} g - \overline{\partial} u$. The function h will only satisfy the equation in the weak sense, but by Weyl's lemma (proved on homework last quarter), this will give $h \in \mathrm{Hol}(\mathbb{C})$ since $h \in C^\infty$.

We will also need $u|_{\mathbb{Z}} = 0$. Solve $\overline{\partial} u = \overline{\partial} g$. If we can solve this equation, then since $\overline{\partial} g \in C^\infty$, we get that $u \in C^\infty(\mathbb{C})$ by Weyl's lemma. By Hörmander's theorem for any $\varphi \in \mathrm{SH}(\mathbb{C})$, there is a solution u such that

$$a \int |u|^2 e^{-\varphi} (1+|z|^2)^{-a} \le \int |\overline{\partial} g|^2 e^{-\varphi} (1+|z|^2)^{2-a} < \infty.$$

Idea (due to Bombieri[12]): choose φ such that $\varphi|_{\mathbb{Z}} = -\infty$ and $e^{-\varphi} \notin L^1$ near $z = k$ for all k, while the right hand side is finite. This will imply that $u(k) = 0$ for all $k \in \mathbb{Z}$. Try:

$$\varphi(z) = 2 \log |\sin(\pi z)| + \log(1+|z|^2).$$

Then

$$e^{-\varphi} \sim \frac{1}{|z-k|^2} \notin L^1 \text{ near } z = k$$

Also take $a = 2$. Check that the right hand side equals

$$\int |\overline{\partial} g|^2 \frac{1}{|\sin(\pi z)|^2} \frac{1}{1+|z|^2} \, L(dz).$$

Since $\overline{\partial} g = \sum b_k \overline{\partial} \psi(z-k)$, $1/|\sin(\pi z)|$ is bounded on the support of $\overline{\partial} g$.

We get that $h = g - u$, which is a holoorphic solution of $h(k) = b_k$ such that

$$\int |u(z)|^2 \frac{1}{|\sin(\pi z)|^2} \frac{1}{(1+|z|^2)^3} < \infty.$$

Since $g \in L^\infty$, we also get

$$\int_{|\operatorname{Im}(z)| \ge 1} |h|^2 \frac{1}{|\sin(\pi z)|^2} \frac{1}{(1+|z|^2)^3} < \infty. \qquad \square$$

[12] This idea came some time after Hörmander's theorem. It was originally for the several complex variable case, but we can use it in this case with no issue.

24.3 Plurisubharmonic functions

We want to prove L^2 estimates for the $\overline{\partial}$ problem in the case of several complex variables. We need to first say what the analogue of a subharmonic function is.

Definition 24.1. Let $\Omega \subseteq \mathbb{C}^n$ be open. A function $u : \Omega \to [-\infty, \infty)$ is called **plurisubharmonic** if

1. u is upper semicontinuous
2. for all $z \in \Omega$ and $w \in \mathbb{C}^n$, the function $\tau \mapsto u(z + \tau w)$ is subharmonic where it is defined.

25 Plurisubharmonic Functions and the $\overline{\partial}$ Problem in Several Complex Variables

25.1 Properties of plurisubharmonic functions

Let $\Omega \subseteq \mathbb{C}^n$ be open. Last time, we said that $u : \Omega \to [-\infty, \infty)$ is plurisubharmonic if

1. u is upper semicontinuous
2. for al $z \in \Omega$ and $w \in \mathbb{C}$, $\mathbb{C} \ni \tau \to u(z + \tau w)$ is subharmonic.

Example 25.1. Let $f \in \mathrm{Hol}(\Omega)$ for an open $\Omega \subseteq \mathbb{C}^n$. Then $\log |f|$ and $|f|^a$ are plirisubharmonic for $a > 0$.

Proposition 25.1. *Let $u \in C^2(\Omega)$ be real. Then u is plurisubharmonic if and only if for any $z \in \Omega$ and $w \in \mathbb{C}^n$,*

$$\sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}(z) w_j \overline{w}_k \geq 0.$$

Proof. We have that u is plurisubharmonic iff $\Delta_\tau(u(z + \tau w)) \geq 0$:

$$\partial_\tau(u(z + \tau w)) = \sum_{j=1}^{n} \frac{\partial u}{\partial z_j}(z + \tau w) w_j.$$

$$\partial_{\overline{\tau}}(\partial_\tau(u(z + \tau w))) = \sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}(z + \tau w) w_j \overline{w}_k \geq 0. \qquad \square$$

Remark 25.1. The Hermitian form $\mathcal{L}_u(w) = u''_{z,\overline{z}} \overline{w} \cdot w \geq 0$ is called the **Levi form** of u.

Plurisubharmonic functions have the following properties:

Proposition 25.2. *If $\Omega \subseteq \mathbb{C}^n$ is connected and $u \not\equiv -\infty$ is plurisubharmonic in Ω, then $u \in L^1_{\mathrm{loc}}$.*

Proof. Use the same argument as for subharmonic functions, using the sub-mean value property. If $n = 2$, let $D = D_1 \times D_2 \subseteq \mathbb{C}^2$ be a polydisc with $D_j = D(z_j^0, r_j)$. Then

$$\iint_D u(z_1, z_2)\, L(d(z_1, z_2)) \geq \int_{D_1} u(z_1, z_2^0)\, dm \geq m(D) u(z_1^0, z_2^0). \qquad \square$$

Proposition 25.3 (Regularization of plurisubharmonic functions)**.** *Let $0 \leq \varphi \in C_0^\infty(\mathbb{C}^n)$ be such that $\int \varphi = 1$ and φ depends only on $|z_1|, \dots, |z_n|$. Then $u_\varepsilon = u * \varphi_\varepsilon \in C^\infty \cap \mathrm{PSH}$, where $\varphi_\varepsilon(z) = \frac{1}{\varepsilon^{2n}} \varphi(z/\varepsilon)$, and $u_\varepsilon \downarrow u$ as $\varepsilon \downarrow 0$.*

25.2 L^2-estimates for the $\overline{\partial}$-operator for several complex variables

Let $\Omega \subseteq \mathbb{C}^n$ be open. We will study $\overline{\partial} u = f$, where $u \in L^2_{\text{loc}}$ and f is a 1-form: $f = \sum f_j \, d\overline{z}_j$.[13] Then

$$\overline{\partial} f = 0 \iff \frac{\partial f_j}{\partial \overline{z}_k} = \frac{\partial f_k}{\overline{\partial} z_j} \qquad \forall j, k, f_j \in L^2_{\text{loc}}$$

in the weak sense.

We will develop a Hilbert space approach to this problem. Let $H_1 = L^2(\Omega, e^{-\varphi_1})$, where $\varphi_1 \in C^\infty(\Omega)$ is real. Let

$$H_2 = L^2_{(0,1)}(\Omega, e^{-\varphi_2}) = \{f = \sum_{j=1}^n f_j \, dz_j : f_j \in L^2(\Omega, e^{-\varphi_2})\}, \qquad \|f\|^2 = \sum \|f_j\|^2_{\varphi_2},$$

where $\varphi_2 \in C^\infty(\Omega)$. Consider the densely defined operator $T : H_1 \to H_2$ sending $u \mapsto \overline{\partial} u$, where

$$D(T) = \{u \in L^2(\Omega, e^{-\varphi_1}) \mid \overline{\partial} u \in L^2_{(0,1)}(\Omega, e^{-\varphi_2}) : \exists f_j \in L^2(\Omega, e^{\varphi_2}) \text{ s.t. } \frac{\partial u}{\partial \overline{z}_j} = f_j \text{ weakly}\}.$$

Definition 25.1. Let H_1, H_2 be Hilbert spaces. A linear map $T : H_1 \to H_2$ with domain $D(T)$ is **closed** if the graph of T, $G(T) = \{x, Tx) : x \in D(T)\} \subseteq H_1 \times H_2$ is closed.

In other words, if $x_n \in D(T)$ is such that $x_n \to x \in H_1$ and $Tx_n \to y$, then $x \in D(T)$, and $y = Tx$.

We have that $T = \overline{\partial} : L^2(\Omega, e^{-\varphi_1}) \to L^2_{(0,1)}(\Omega, e^{-\varphi_2})$ is closed. We have that $\text{Ran}(T) \subseteq F = \{f \in L^2_{(0,1)}(\Omega, e^{-\varphi_2}) : \overline{\partial} f = 0 \text{ weakly}\} \subseteq H_2$, a closed subspace. We will try to show that $\text{Ran}(T) = F$ for suitable φ_1, φ_2. Introduce the adjoint of T:

Definition 25.2. Let $T : H_1 \to H_2$ be linear and densely defined. We define the **adjoint** $T^* : H_2 \to H_1$ as follows:

$$D(T^*) = \{v \in H_2 : \exists f \in H_1 \text{ s.t. } \langle Tu, v \rangle_{H_2} = \langle u, f \rangle_{H_1} \ \forall u \in D(T)\}.$$

We let $T^* c = f$ ($D(T)$ is dense, so f is unique).

Remark 25.2. Like T itself, the adjoint may be unbounded.

Proposition 25.4. *The adjoint satisfies the following property:*

1. *If T is closed and densely defined, then T^* is closed and densely defined.*

2. *$T^{**} = T$.*

[13] This is sometimes called a $(0,1)$-form, as it has no z_j differentials.

Example 25.2. Let $H_1 = L^2(\Omega, e^{-\varphi_1})$, $H_2 = L^2_{(0,1)}(\Omega, e^{-\varphi_2})$. and $T = \overline{\partial}$. Then

$$D(\overline{\partial}^*) = \{v \in L^2_{(0,1)}(\Omega, e^{-\varphi_2}) : \forall u \in D(\overline{\partial}), \sum_{j=1}^n \int_\Omega \frac{\partial u}{\partial \overline{z}_j} \overline{v}_j e^{-\varphi_2} \, L(dz) = \int_\Omega u \overline{f} e^{-\varphi_1} \, L(dz)$$
$$\text{for some } f \in L^2(\Omega, e^{-\varphi_1})\}.$$

By integration by parts, $C^\infty_{0,(0,1)}(\Omega) \subseteq D(\overline{\partial}^*)$. If $v \in D(\overline{\partial}^*)$, we get taking $u \in C_0^\infty$ that $f = \overline{\partial}^* v = -\sum_{j=1}^n e^{\varphi_1} \partial_{z_j}(e^{-\varphi_2} v_j)$, where these are weak derivatives.

We have a closed $T : H_1 \to H_2$ where $\operatorname{Ran}(T) \subseteq F \subseteq H_2$ is closed. Next time, we will show the following.

Lemma 25.1. $\operatorname{Ran}(T) = F$ *if and only if there exists* $C > 0$ *such that* $\|f\|_{H_2} \leq C \|T^* f\|_{H_1}$ *for all* $f \in F \cap D(T^*)$.

26 L^2 Estimates for The $\overline{\partial}$ Operator in Several Complex Variables (cont.)

26.1 Conditions for an operator to be surjective

We have an operator $T : L^2(\Omega, e^{-\varphi_1}) \to L^2_{(0,1)}(\Omega, e^{-\varphi_2})$, acting as $\overline{\partial}$, where $\Omega \subseteq \mathbb{C}^n$ is open and $\varphi_1, \varphi_2 \in C^\infty(\Omega)$ are real weights to be chosen. Also $\mathrm{Ran}(T) \subseteq F = \{f \in L^2_{(0,1)}(\Omega, e^{-\varphi_2}) : \overline{\partial} f = 0\}$.

Lemma 26.1. *Let $T : H_1 \to H_2$ be linear, closed, and densely defined with $\mathrm{Ran}(T) \subseteq F$, where F is a closed subspace of H_2. Then $\mathrm{Ran}(T) = F$ if and only if there is a $C > 0$ such that $\|F\|_{H_2} \le C\|T^* f\|_{H_1}$ for all $f \in F \cap D(T^*)$.*

Proof. ($\Longrightarrow$): Consider the map $T : D(T) \to F$, which are Banach spaces if $D(T)$ is equipped with the graph norm $\|u\|_{D(T)} := \|u\| + \|Tu\|$. T is continuous and surjective, so T is open by the open mapping theorem. Then $T(\{u : \|u\|_{D(T)} < 1\}) \supseteq \{f \in F : \|f\| < \varepsilon\}$ for some $\varepsilon > 0$. We get that there is a $C > 0$ such that for all $g \in F$, there is a $u \in D(T)$ such that $Tu = f$ and $\|u\|_{H_1} \le C\|g\|_{H_2}$. When $f \in D(T^*) \cap F$,

$$|\langle f, g\rangle_{H_2}| = |\langle f, Tu\rangle_{H_2}| = |\langle T^* f, u\rangle| \le C\|T^* f\|_{H_1}\|g\|_{H_2}.$$

We get that $\|f\|_{H_2} \le \|T^* f\|_{H_1}$.

($\Longleftarrow$): Assume that the bound holds for all $f \in F \cap D(T^*)$. We have $\mathrm{Ran}(T) \subseteq F$. Let $g \in F$. We claim that the antilinear map $L(T^* f) = \langle f, g\rangle_{H_2}$ (for $f \in D(T^*)$) is well-defined and satisfies $|L(T^* f)| \le C\|g\|_{H_2}\|T^* f\|_{H_1}$.

We can write $f = f_! + f_2$, where $f_1 \in F$, and $f_2 \in F^\perp$ for any $f \in D(T^*)$. Now $\langle f_2, Tu\rangle = 0$ for any $u \in D(T)$, so $f_2 \in D(T^*)$; in particular, $T^* f_2 = 0$. So $f_1 \in F \cap D(T^*)$, and we get

$$|L(T^* f)| = \langle g, f_1\rangle \le C\|g\|_{H_2}\|\underbrace{T^* f_1}_{=T^* f}\|_{H_1}.$$

So we get the claim.

We get that the map L extends by continuity to $\overline{\mathrm{Ran}(T^*)} \subseteq H_1$. So there is a $u \in \overline{\mathrm{Ran}(T^*)}$ such that $L(T^* f) = \langle u, T^* f\rangle_{H_1}$ for all $f \in D(T^*)$. On the other hand, $L(T^* f) := \langle g, f\rangle_{H_2}$, so we get $\langle T^* f, u\rangle = \langle f, g\rangle$ for all $f \in D(T^*)$. This implies that $u \in D((T^*)^*) = D(T)$ and $Tu = g$. We also get that

$$\|u\|_{H_1} = \|L\| \le C\|g\|_{H_2}. \qquad \square$$

26.2 Hörmander's idea and the density lemma

In our setting $H_1 = L^2(\Omega, e^{-\varphi_1})$, $H_2 = L^2_{(0,1)}(\Omega, e^{-\varphi_2})$, $T = \overline{\partial}$, and $F = \{f \in H_2 : \overline{\partial} f = 0\}$. So we want to show that

$$\|f\|_{H_2} \le C\|T^* f\|_{H_1}, \qquad f \in F \cap D(T^*).$$

Introduce the space of 2-forms

$$H_3 = L^2_{(0,2)}(\Omega, e^{-\varphi_3}) = F = \sum_{j,k} F_{j,k}\, d\bar{z}_j \wedge d\bar{z}_k : F_{j,k} \in L^2(\Omega, e^{-\varphi_3}),$$

and consider the closed, densely defined operator $S : H_2 \to H_3$ which sends $f \mapsto \overline{\partial} f = \sum_j \overline{\partial} f_j \wedge d\bar{z}_j = \sum_{j,k} \frac{\partial f_j}{\partial \bar{z}_k} d\bar{z}_k \wedge d\bar{z}_h$. We have $F = \ker(S)$. Rather than trying to prove the bound, we shall try to prove

$$\|f\|^2_{H_2} \leq C(\|T^* f\|^2_{H_1} + \|Sf\|^2_{H_3}), \qquad \forall f \in D(T^*) \cap D(S).$$

This looks stronger, but it has symmetry properties we can exploit.

The idea, due to Hörmander, is to choose the weights $\varphi_1, \varphi_2, \varphi_3$ so that the 1-forms with coefficients in $C_0^\infty(\Omega)$ are dense with respect to the graph norm $f \mapsto \|f\|_{H_2} + \|T^* f\|_{H_1} + \|Sf\|_{H_3}$.

Lemma 26.2 (Density lemma)**.** *Let (η_ν) be a sequence in $C_0^\infty(\Omega)$ such that $0 \leq \eta_\nu \leq 1$ and such that for any compact $K \subseteq \Omega$, $\eta_\nu = 1$ on K for all large ν. Assume that*

$$e^{-\varphi_{j+1}} |\overline{\partial} \eta_\nu|^2 \leq C e^{-\varphi_j}, \qquad \forall \nu, j = 1, 2.$$

Then $C^\infty_{0,(0,1)}(\Omega)$ is dense in $D(T^*) \cap D(S)$ with respect to the graph norm.

Remark 26.1. If $\Omega = \mathbb{C}^n$, we can take $\eta_\nu(z) = \eta(z/\nu)$ for some function η which is 1 near 0. Then we can take $\varphi_1 = \varphi_2 = \varphi_3$.

Proof. Step 1: Suppose $f \in D(T^*) \cap D(S)$ has compact support. Approximate by $f * \psi_\varepsilon$, where $\psi_\varepsilon(z) = \varepsilon^{-2n} \psi(z/\varepsilon)$ and $\psi \in C_0^\infty$.

Step 2: Given $f \in D(T^*) \cap D(S)$, consider $\eta_\nu f \in D(T^*) \cap D(S)$. Then $S(\eta_j f) \to Sf$ in H_3. Then

$$S(\eta_j f) = \underbrace{\eta_j S f}_{L^2_{\varphi_3}} + \underbrace{[S, \eta_j]}_{= (\overline{\partial} \eta_j) f} \xrightarrow{L^2_{\varphi_3}} Sf$$

by dominated convergence. □

We will review this last point in more detail next time.

27 L^2-Estimates for the $\overline{\partial}$-Operator: The Density Lemma

27.1 The density lemma

In solving our $\overline{\partial}$ problem, we have

$$L^2(\Omega, e^{-\varphi_1}) \xrightarrow{T} L^2_{(0,1)}(\Omega, e^{-\varphi_2}) \xrightarrow{S} L^2_{(0,2)}(\Omega, e^{-\varphi_3}).$$

We want to show that

$$\|f\|_{\varphi_2} \leq C(\|T^*f\|^2_{\varphi_1} + \|Sf\|^2_{\varphi_3}), \qquad \forall f \in D(T^*) \cap D(S).$$

We had the following lemma:

Lemma 27.1 (Density lemma)**.** *Let (η_ν) be a sequence in $C_0^\infty(\Omega)$ such that $0 \leq \eta_\nu \leq 1$ and such that for any compact $K \subseteq \Omega$, $\eta_\nu = 1$ on K for all large ν. Assume that*

$$e^{-\varphi_{j+1}}|\overline{\partial}\eta_\nu|^2 \leq Ce^{-\varphi_j}, \qquad \forall \nu, j = 1, 2.$$

Then $C^\infty_{0,(0,1)}(\Omega)$ is dense in $D(T^*) \cap D(S)$ with respect to the graph norm.

Proof. Step 1: Suppose $f \in D(T^*) \cap D(S)$ has compact support. Approximate by $f * \psi_\varepsilon$, where $\psi_\varepsilon(z) = \varepsilon^{-2n}\psi(z/\varepsilon)$ and $\psi \in C_0^\infty$.

Step 2: Let $f \in D(T^*) \cap D(S)$. We claim that $\eta_j f \in D(T^*) \cap D(S)$. To show that $\eta_j f \in D(S)$,

$$\overline{\partial}(\eta_j f) = \underbrace{\eta_j \underbrace{\overline{\partial} f}_{\in L^2_{\varphi_3}}}_{\in L^2_{\varphi_3}} + \underbrace{\overline{\partial}\eta_j \wedge f}_{\in L^2_{\varphi_3}}.$$

To show that $\eta_j f \in D(T^*)$, consider for $u \in D(T)$,

$$\langle Tu, \eta_j f\rangle_{\varphi_2} = \langle \eta_j Tu, f\rangle_{\varphi_2}$$

Observe that $\eta_j Tu = \eta_j \overline{\partial} u = \overline{\partial}(\eta_j u) - u\overline{\partial}\eta_j$, where $\eta_j u \in D(T)$.

$$\begin{aligned} &= \langle T(\eta_j u), f\rangle_{\varphi_2} - \int u \left\langle \overline{\partial}\eta, f\right\rangle e^{-\varphi_2} \\ &= \langle u, \eta_j T^* f\rangle_{\varphi_1} - \left\langle u, e^{\varphi_1 - \varphi_2} \left\langle \overline{\partial}\eta, f\right\rangle\right\rangle_{\varphi_1}. \end{aligned}$$

So

$$T^*(\eta_j f) = \eta_j T^* f - e^{-\varphi_1 - \varphi_2}\left\langle \overline{\partial}\eta, f\right\rangle.$$

We now check that $\eta_j f \to f$ in the graph norm.

1. $\eta_j f \to f$ in $L^2_{\varphi_2}$: This follows by the dominated convergence theorem.

2. $S(\eta_j f) \to Sf$ in L_{φ_3}: We have

$$S(\eta_j f) = \overline{\partial}(\eta_j f) = \underbrace{\eta_j \underbrace{Sf}_{\in L^2_{\varphi_3}}}_{\to Sf \text{ in } L^2_{\varphi_3}} + \underbrace{\overline{\partial}\eta_j \wedge f}_{\to 0 \text{ in } L^2_{\varphi_3}}$$

So we get that

$$\int \underbrace{|\overline{\partial}\eta_j|^2 e^{-\varphi_3}}_{\le e^{-\varphi_2}} |f|^2 \to 0$$

by the dominated convergence theorem.

3. $T^*(\eta_j f) \to T^* f$ in $L^2_{\varphi_1}$ is similar. □

27.2 Applying the lemma

Now let $\psi \in C^\infty(\Omega)$ be given by the locally finite sum

$$e^\psi = 1 + \sum_{\nu=1}^\infty |\overline{\partial}\eta_\nu|^2.$$

Let $\varphi_j = \varphi + (j-3)\psi$ for $j = 1, 2, 3$ (φ is to be chosen). With this choice of weights, we can satisfy the hypotheses of the density lemma.

We will now study our estimate

$$\|f\|^2_{\varphi_2} \le C(\|T^* f\|^@_{\varphi_1} + \|Sf\|^2_{\varphi_2}), \qquad f \in C_0^\infty.$$

Recall the formula for T^*:

$$T^* f = -e^{\varphi_1} \sum_{j=1}^\infty \partial_{z_j}(f_j e^{-\varphi_2}) = -e^{\varphi - 2\psi} \sum_{j=1}^\infty \partial_{z_j}(f_j e^{\psi - \varphi}).$$

Then

$$e^\psi T^* f = -\sum \delta_j f_j - \sum f_j \partial_{z_j}\psi, \qquad \delta_j := \partial_{z_j} - \partial_{z_j}\varphi.$$

Here, $-\delta_j$ is the adjoint of $\partial_{\overline{z}_j}$ in L^2_φ.

Consider

$$\|T^* f\|^2_{\varphi_1} = \int |T^* f|^2 e^{-\varphi + 2\psi} = \|e^\psi T^* f\|_\varphi.$$

Then, using Cauchy-Schwarz or the triangle inequality,

$$\left\|\sum \delta_j f_j\right\|^2_\varphi = \|e^\psi T^* f + \langle f, \partial\psi\rangle\|^2_\varphi$$

$$\leq 2\|T^*f\|_{\varphi_1}^2 + 2\int |\langle t, \partial\psi\rangle|^2 e^{-\varphi}.$$

Compute $\|Sf\|_{\varphi_3}^2$:

$$Sf = \overline{\partial} f = \sum_{j<k} \left(\frac{\partial d_k}{\partial \overline{z}_j} - \frac{\partial f_j}{\partial \overline{z}_k} \right) d\overline{z}_j \wedge d\overline{z}_k.$$

So

$$\begin{aligned}
\|Sf\|_{\varphi_3}^2 &= \sum_{j<k} \int \left| \frac{\partial f_k}{\partial \overline{z}_j} - \frac{\partial f_j}{\partial \overline{z}_k} \right|^2 e^{-\varphi} \\
&= \frac{1}{2} \sum_{j,k} \int \left| \frac{\partial f_k}{\partial \overline{z}_j} - \frac{\partial f_j}{\partial \overline{z}_k} \right|^2 e^{-\varphi} \\
&= \int \sum_{j,k} \left| \frac{\partial f_k}{\partial \overline{z}_j} \right|^2 e^{-\varphi} - \left(\sum_{j,k} \frac{\partial f_j}{\partial \overline{z}_k} \overline{\frac{\partial f_k}{\partial \overline{z}_j}} \right) e^{-\varphi}
\end{aligned}$$

Add $\|Sf\|_{\varphi_3}^2$ to both sides of the inequality. We get the following estimate:

$$\left\| \sum \delta_j f_j \right\|_{\varphi}^2 - \sum_{j,k} \left\langle \partial_{\overline{z}_k} f_j, \partial_{\overline{z}_j} f_k \right\rangle_{\varphi} \leq 2\|T^*f\|_{\varphi_1}^2 + 2\int |\langle f, \partial\psi\rangle|^2 e^{-\varphi} + \|Sf\|_{\varphi_3}^2.$$

The main point of the argument is that

$$\begin{aligned}
\langle \delta_j f_j, \delta_k, f_k \rangle_{\varphi} - \left\langle \partial_{\overline{z}_k} f_j, \partial_{\overline{z}_j} f_k \right\rangle_{\varphi} &= - \langle \partial_{\overline{z}_k} \delta_j f_j, f_k \rangle_{\varphi} + \left\langle \delta_{z_j} \partial_{\overline{z}_k} f_j, f_k \right\rangle_{\varphi} \\
&= \left\langle [\delta_{z_j}, \partial_{\overline{z}_k}] f_j, f_k \right\rangle_{\varphi}.
\end{aligned}$$

The commutator equals

$$[\partial_{z_j} - \partial_{z_j}\varphi, \partial_{\overline{z}_k}] = \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k}.$$

So the lower bound becomes

$$\int \sum_{j,k} \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k} f_j f_k e^{-\varphi},$$

where $\frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k}$ is the Levi form of $\varphi(f)$. Now we can choose φ to be plurisubharmonic.

We will conclude our discussion next time.

28 L^2-Estimates for the $\overline{\partial}$-Operator

28.1 Solution of the $\overline{\partial}$ problem

Recall that

$$\sum_{j,k=1}^{n} \int_\Omega \frac{\partial\varphi}{\partial z_j \partial \overline{z}_k} f_j \overline{f}_k e^{-\varphi} \, L(dz) \leq 2 \int |f|^2 |\partial\psi|^2 e^{-\varphi} + 2\|T^* f\|_{\varphi_1}^2 + \|Sf\|_{\varphi_3}^2,$$

where $f \in C^\infty_{0,(0,1)}(\Omega)$, $\Omega \subseteq \mathbb{C}^n$ is open, $\varphi_1 = \varphi - 2\psi$, and $\varphi_3 = \varphi$. Assume that $\varphi \in C^\infty(\Omega)$ is **strictly plurisubharmonic**: there exists $0 < c(z) \in C(\Omega)$ such that

$$\sum_{j,k=1}^{n} \frac{\partial^2\varphi}{\partial z_j \partial \overline{z}_k} w_j \overline{w}_k \geq c(z)|w|^2, \qquad z \in \Omega, w \in \mathbb{C}^n.$$

First consider the simplest case, $\Omega = \mathbb{C}^n$. We can then take $\psi = 0$, and it follows that

$$\int c|f|^2 e^{-\varphi} \leq \|T^* f\|_\varphi^2 + \|Sf\|_\varphi, \qquad f \in C^\infty_{0,(0,1)}(\mathbb{C}^n).$$

Recall that $T = \overline{\partial} : L^2(\mathbb{C}^n, e^{-\varphi}) \to L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi})$ and $S = \overline{\partial} : L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi}) \to L^2_{(0,2)}(\mathbb{C}^n, e^{-\varphi})$ are closed and densely defined with natural domains, By the density lemma, this inequality extends to all $f \in D(T^*) \cap D(S)$.

Theorem 28.1. *Let $\varphi \in C^\infty(\mathbb{C}^n)$ be strictly plurisubharmonic with*

$$\sum_{j,k=1}^{n} \frac{\partial^2\varphi}{\partial z_j \partial \overline{z}_k} w_j \overline{w}_k \geq c(z)|w|^2, \qquad 0 < c \in C(\mathbb{C}^n).$$

Then for all $g \in L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi})$ with $\partial g = 0$ and $\int |g|^2/ce^{-\varphi} < \infty$, there exists some $u \in L^2(\mathbb{C}^n, e^{-\varphi})$ such that $\overline{\partial} u = g$ and

$$\int |u|^2 e^{-\varphi} \leq \int \frac{|g|^2}{c}.$$

Proof. We must solve the equation $Tu = g$ so that the above conclusion holds. Note that

$$\begin{aligned} Tu = g &\iff \forall f \in D(T^*), \langle Tu, f\rangle_\varphi = \langle g, f\rangle \qquad (D(T^*) \text{ is dense}) \\ &\iff \langle u, T^* f\rangle_\varphi = \langle g, f\rangle_\varphi \ \forall f \in D(T^*) \qquad (T \text{ is closed}). \end{aligned}$$

We claim that

$$|\langle g, f\rangle_\varphi| \leq \|T^* f\|_\varphi \left(\int \frac{|g|^2}{c} e^{-\varphi} \right)^{1/2}, \qquad f \in D(T^*).$$

Indeed, if f is orthogonal to $\ker(S) \ni g$, then the left hand side equals 0. Also, $\operatorname{ran}(T) \subseteq \ker(S)$, so if $\langle f, Tu \rangle_\varphi = 0$ for all $u \in D(T)$, then $f \in D(T^*)$ and $T^* f = 0$; so the right hand side equals 0. If $f \in D(T*) \cap \ker(S)$, we get (by Cauchy-Schwarz) that

$$
\begin{aligned}
|\langle g, f \rangle_\varphi|^2 &= \left| \int \langle g, \overline{f} \rangle e^{-\varphi} \right|^2 \\
&\leq \left(\int c|f|^2 e^{-\varphi} \right) \int \frac{|g|^2}{c} e^{-\varphi} \\
&\leq \|T^* f\|_\varphi^2 \int \frac{|g|^2}{c} e^{-\varphi}.
\end{aligned}
$$

The claim follows, and the antilinear form $T^* f \mapsto \langle g, f \rangle_\varphi$ for $f \in D(T^*)$ extends to a continuous linear form on $L^2(\mathbb{C}^n, e^{-\varphi})$ with norm $\leq \left(\int \frac{|g|^2}{c} e^{-\varphi} \right)^{1/2}$.

So there exists some $u \in L^2(\mathbb{C}^n, e^{-\varphi})$ with $\|u\|_\varphi^2 \leq \int \frac{|g|^2}{c} e^{-\varphi}$ and $\langle g, f \rangle_\varphi = \langle u, T^* f \rangle$ for all $f \in D(T^*)$. So $u \in D(T)$, and $Tu = g$. $\square$

28.2 Extensions

Arguing as in the 1 dimensional case, replacing φ by $\varphi + 2\log(1 + |z|^2)$ (the latter term is strictly plurisubharmonic on $\mathbb{C}^n$) and regularizing φ, we get the following result:

Theorem 28.2. *Let $\varphi \in \mathrm{PSH}(\mathbb{C}^n)$ with $\varphi \not\equiv -\infty$. For all $g \in L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi})$ such that $\overline{g} = 0$, there exists a $u \in L^2_{\mathrm{loc}}(\mathbb{C}^n, e^{-\varphi})$ such that $\overline{\partial} u = g$ and*

$$
2 \int |u|^2 e^{-\varphi} (1 + |z|^2)^{-2} \leq \int |g|^2 e^{-\varphi}.
$$

Remark 28.1. There exist analogous results when $\mathbb{C}^n$ is replaced by an open set $\Omega \subseteq \mathbb{C}^n$, provided that Ω is **pseudoconvex**: there exists $u \in C(\Omega) \cap \mathrm{PSH}(\Omega)$ such that for all $t \in \mathbb{R}$, the set $\{z \in \Omega : u(z) < t\}$ is relatively compact in Ω. (Notice that any open set $\Omega \subseteq \mathbb{C}$ is pseudoconvex.)